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An extension of skeleton by influence zones and morphological interpolation to fuzzy sets

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Abstract While the notions of influence zones and skeleton by influence zones (SKIZ) have many important applications, they have not been extended to fuzzy sets until now. The aim of this paper is to fill this gap and to show the potential usefulness of such an extension. The proposed definitions are based on fuzzy dilations and their interpretations in terms of distances. As another contribution, we show how this notion can be used to define a fuzzy median set, and a series of fuzzy sets interpolating between two fuzzy sets.

Keywords: fuzzy sets, fuzzy skeleton by influence zones, fuzzy median set, interpolation between fuzzy sets.

1. Introduction

Despite the interest of notions of influence zones and skeleton by influence zones (SKIZ), surprisingly enough they have not really been exploited in a fuzzy context until now. If knowledge or information is modeled using fuzzy sets, it is natural to see the influence zones of these sets as fuzzy sets too. The extension of these notions to the fuzzy case is therefore important, for applications such as partioning the space where fuzzy sets are defined, implementing the notion of separation, reasoning on fuzzy sets (fusion, interpolation, negotiations, spatial reasoning on fuzzy regions of space, etc.), motivating the work presented in this paper.

The first contribution is to propose definitions of notions of influence zones and skeleton by influence zones for fuzzy sets. Both influence zones and the SKIZ are then fuzzy sets, defined on the same space. The proposed definitions rely on formal expressions of the SKIZ in terms of distances and morphological dilations. Let $\mathcal{S}$ be the underlying space, endowed with a distance d, and X be a subset of $\mathcal{S}$ composed of several connected components: $X = \bigcup_i X_i$, with $X_i \cap X_j = \emptyset$ for $i \neq j$. The influence zone of X_i, denoted as $IZ(X_i)$, is defined as [15,18]:

$$IZ(X_i) = \{x \in \mathcal{S} \ / \ d(x, X_i) < d(x, X \setminus X_i)\}. \quad (1)$$

The SKIZ of X, denoted as SKIZ(X), is then given by:

$$\text{SKIZ}(X) = (\textstyle\bigcup_i IZ(X_i))^c.$$

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Let us denote by δ_λ the dilation by a ball of radius λ, and ε_λ the erosion by a ball of radius λ. Then the influence zones can be expressed as:

$$IZ(X_i) = \bigcup_\lambda (\delta_\lambda(X_i) \cap \varepsilon_\lambda((\cup_{j\neq i} X_j)^c)) = \bigcup_\lambda (\delta_\lambda(X_i) \setminus \delta_\lambda(\cup_{j\neq i} X_j)). \quad (2)$$

These two expressions of influence zones, in terms of morphological dilations on the one hand and in terms of distances on the other hand, constitute the basis for the proposed definitions in the fuzzy case (Section 2).

The second contribution (Section 3) is to exploit the notion of fuzzy SKIZ to define the median fuzzy set of two intersecting fuzzy sets. The iterative application of the median set computation leads to the construction of a series of interpolating sets from one fuzzy set to another one. To our knowledge, this idea of interpolation between fuzzy sets is also novel.

2. Fuzzy influence zones and fuzzy SKIZ

While several notions involved in the SKIZ definition have been generalized to fuzzy sets (such as distances, dilations, erosions) influence zones and SKIZ have, to the best of our knowledge, never been defined in the case of fuzzy sets. This is the aim of this section. We consider two fuzzy sets, with membership functions μ_1 and μ_2 defined on $\mathcal{S}$. The extension to an arbitrary number of fuzzy sets is then straightforward.

2.1 Fuzzy structuring element and fuzzy dilation and erosion

The morphological operations involved in the crisp case are performed using a structuring element which is a ball of a distance. In $\mathbb{R}^n$, the Euclidean distance is generally considered. In a digital space, such as $\mathbb{Z}^n$, a discrete distance is defined, based on an underlying discrete connectivity. The ball of radius 1 of this distance is then constituted by the center point and its neighbors according to the choice of the connectivity. More generally, the structuring element can be defined from a binary relation on $\mathcal{S}$, that is assumed to be symmetrical in this paper (which is consistent if it is a ball of a distance). In the fuzzy case, the same crisp structuring elements can be used. We can also base the operations on a fuzzy structuring element, which can represent local imprecision or a fuzzy binary relation. We denote the structuring element by its membership function ν. All what follows applies for crisp and for fuzzy structuring elements. In $\mathbb{R}^n$ or $\mathbb{Z}^n$, $\nu(x)$ represents the degree to which x belongs to ν and $\nu(y-x)$ the degree to which y belongs to the translation of ν at point x. If ν is derived from a fuzzy binary relation, $\nu(y-x)$ denotes the degree to which y is in relation to x.

Let us denote by $\delta_\nu(\mu)$ and $\varepsilon_\nu(\mu)$ the dilation and erosion of the fuzzy set μ by the structuring element ν. Here, dual definitions of these operations are chosen [5], i.e. verifying $\varepsilon_\nu(\mu^c) = (\delta_\nu(\mu))^c$, since this property is important,

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as seen in Equation 2. They are expressed as:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} \top[\mu(y), \nu(y-x)], \tag{3}$$

$$\forall x \in \mathcal{S},\ \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} \bot[\mu(y), c(\nu(y-x))], \tag{4}$$

where $\top$ is a t-norm and $\bot$ the t-conorm dual of $\top$ with respect to a complementation c (which automatically guarantees the duality between δ and ε). Examples of t-norms are min, product, Lukasiewicz $(\max(0, a+b-1))$, and they generalize intersection to fuzzy sets, while t-conorms generalize union (examples are max, algebraic sum and Lukasiewicz $\min(1, a+b)$). In this paper, the following classical complementation is used: $\forall t \in [0,1], c(t) = 1-t$. Other definitions of fuzzy mathematical morphology have been proposed (e.g. [7, 8, 14]), based on different operators. Links with the ones used here are developed in [3, 5].

Important properties of the definitions given in Equations 3 and 4, that will be intensively used in the following, are: (i) fuzzy dilation and erosion are equivalent to the classical dilation and erosion in case both μ and ν are crisp; (ii) $\nu(0) = 1 \Rightarrow \mu \leq \delta_\nu(\mu)$ and $\varepsilon_\nu(\mu) \leq \mu$, where 0 denotes the origin of $\mathcal{S}$ (if ν represents a binary relation, it means that this relation is reflexive); (iii) fuzzy dilation and erosion are increasing with respect to μ, dilation is increasing with respect to ν while erosion is decreasing; (iv) fuzzy dilation commutes with the supremum and fuzzy erosion with the infimum; (v) duality: $\varepsilon_\nu(\mu^c) = (\delta_\nu(\mu))^c$; (vi) iterativity property: successive dilations (respectively erosions) are equivalent to one dilation (respectively erosion) with a structuring element equal to the dilation of all structuring elements.

2.2 Definition based on fuzzy dilations

Let us first consider the expression of influence zone using morphological dilations (Equation 2). This expression can be extended to fuzzy sets by using fuzzy intersection and union, and fuzzy mathematical morphology.

Definition 1. For a given structuring element ν, we define the influence zone of μ_1 as:

$$IZ_{dil}(\mu_1) = \bigcup_\lambda (\delta_{\lambda\nu}(\mu_1) \cap \varepsilon_{\lambda\nu}(\mu_2^c)) = \bigcup_\lambda (\delta_{\lambda\nu}(\mu_1) \setminus \delta_{\lambda\nu}(\mu_2)). \tag{5}$$

The dilation by $\lambda\nu$ is obtained by λ iterations of a dilation by ν in the discrete case. The influence zone for μ_2 is defined in a similar way. The extension to any number of fuzzy sets μ_i is straightforward: $IZ_{dil}(\mu_i) = \bigcup_\lambda (\delta_{\lambda\nu}(\mu_i) \cap \varepsilon_{\lambda\nu}((\cup_{j\neq i}\mu_j)^c))$. □

In these equations, intersection and union of fuzzy sets are implemented as t-norms $\top$ and t-conorms $\bot$ (min and max for instance). Equation 5 then reads: $IZ_{dil}(\mu_1) = \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1), 1 - \delta_{\lambda\nu}(\mu_2)]$.

Note that the number of dilations to be performed to compute influence zones in a digital bounded space $\mathcal{S}$ is always finite (and bounded by the length of the largest diagonal of $\mathcal{S}$).

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Definition 2. The fuzzy SKIZ is then defined as:

$$\mathrm{SKIZ}(\cup_i \mu_i) = (\bigcup_i IZ(\mu_i))^c.$$

□

This expression also defines a fuzzy (generalized) Voronoï diagram. Although the notion of Voronoï diagram has already been used in fuzzy systems, to our knowledge, no fuzzy version of it was defined until now.

2.3 Definitions based on distances

Another approach consists in extending the definition in terms of distances (Equation 1) and defining a degree to which the distance to one of the sets is lower than the distance to the other sets. Several definitions of the distance of a point to a fuzzy set have been proposed in the literature. Some of them provide real numbers and Equation 1 can then be applied directly. But then the imprecision in the object definition is lost. Definitions providing fuzzy numbers are therefore more interesting, since if the sets are imprecise, it may be expected that distances are imprecise too, as also underlined e.g. in [2, 10]. In particular, as will be seen next, it may be interesting to use the distance proposed in [2], based on fuzzy dilation:

$$d(x,\mu)(n) = \top[\delta_{n\nu}(\mu)(x), 1 - \delta_{(n-1)\nu}(\mu)(x)]. \tag{6}$$

It expresses, in the digital case, the degree to which x is at a distance n of μ ($\top$ is a t-norm, and $n \in \mathbb{N}^*$). For $n = 0$, the degree becomes $d(x,\mu)(0) = \mu(x)$. This expression can be generalized to the continuous case as:

$$d(x,\mu)(\lambda) = \inf_{\lambda'<\lambda} \top[\delta_{\lambda\nu}(\mu)(x), 1 - \delta_{\lambda'\nu}(\mu)(x)], \tag{7}$$

where $\lambda \in \mathbb{R}^{+*}$, and $d(x,\mu)(0) = \mu(x)$.

First method: comparing fuzzy numbers

When distances are fuzzy numbers, the fact that $d(x,\mu_1)$ is lower than $d(x,\mu_2)$ becomes a matter of degree. The degree to which this relation is satisfied can be performed using methods for comparing fuzzy numbers. Let us consider the definition in [9], which expresses the degree $\mu(d_1 < d_2)$ to which $d_1 < d_2$, d_1 and d_2 being two fuzzy numbers, using the extension principle:

$$\mu(d_1 < d_2) = \sup_{a<b} \min(d_1(a), d_2(b)). \tag{8}$$

Definition 3. The influence zone of μ_1 based on the comparison of fuzzy numbers (using Equation 8) is defined as:

$$IZ_{dist1}(\mu_1)(x) = \mu(d(x,\mu_1) < d(x,\mu_2)) = \sup_{n<n'} \min[d(x,\mu_1)(n), d(x,\mu_2)(n')]. \tag{9}$$

□

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Note that this approach can be applied whatever the chosen definition of fuzzy distances.

Second method: direct approach

When distances are more specifically derived from a dilation, as the ones in Equations 6 and 7, a more direct approach can be proposed, taking into account explicitly this link between distances and dilations. Indeed, in the binary case, the following equivalences hold:

$$\begin{aligned}(d(x, X_1) \leq d(x, X_2)) &\Leftrightarrow (\forall \lambda, x \in \delta_\lambda(X_2) \Rightarrow x \in \delta_\lambda(X_1)) \\ &\Leftrightarrow (\forall \lambda, x \in \delta_\lambda(X_1) \vee x \notin \delta_\lambda(X_2)). \quad (10)\end{aligned}$$

This expression extends to the fuzzy case as follows.

Definition 4. The degree $\mu(d(x, \mu_1) \leq d(x, \mu_2))$ to which $d(x, \mu_1)$ is less than $d(x, \mu_2)$ is defined as:

$$\mu(d(x, \mu_1) \leq d(x, \mu_2)) = \inf_\lambda \bot(\delta_{\lambda\nu}(\mu_1)(x), 1 - \delta_{\lambda\nu}(\mu_2)(x)), \quad (11)$$

where $\bot$ is a t-conorm. □

This equation defines a new way to compare fuzzy numbers representing distances.

The comparison of fuzzy numbers representing distances, as given by Equation 11 is reflexive ($\mu(d(x, \mu_1) \leq d(x, \mu_1)) = 1$) if and only if $\bot$ is a t-conorm verifying the excluded middle law (Lukasiewicz t-conorm for instance). Moreover, in case the fuzzy numbers are usual numbers, the comparison reduces to the classical comparison between numbers.

Defining influence zones requires a strict inequality between distances, which is deduced by complementation:

$$\mu(d(x, \mu_1) < d(x, \mu_2)) = 1 - \mu(d(x, \mu_2) \leq d(x, \mu_1)). \quad (12)$$

Definition 5. The influence zone of μ_1 using the comparison introduced in Definition 4 is defined as:

$$IZ_{dist2}(\mu_1)(x) = 1 - \inf_\lambda \bot(\delta_{\lambda\nu}(\mu_2)(x), 1 - \delta_{\lambda\nu}(\mu_1)(x)). \quad (13)$$

□

Whatever the chosen definition of IZ, the SKIZ is always defined as in Definition 2.

2.4 Comparison and properties

Proposition 1. *Definitions 1 and 5 are equivalent:* $IZ_{dil}(\mu_1) = IZ_{dist2}(\mu_1)$.

Although this result is not surprising, both interpretations in terms of dilation and distance remain interesting.

However, the two distance based approaches are not equivalent, since they rely on different orderings between fuzzy sets. Actually the direct approach always provides a larger result.

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Proposition 2. $\forall x \in \mathcal{S}, IZ_{dist1}(\mu_1)(x) \leq IZ_{dist2}(\mu_1)(x)$.

Proposition 3. *For N being the size of $\mathcal{S}$ in each dimension, the complexity of computation of IZ_{dist1} is in $O(N^5)$ in 3D and $O(N^4)$ in 2D. The complexity of computation of IZ_{dist2} or IZ_{dil} is at least one order of magnitude less.*

Proposition 4. *Definitions 1, 2, 3 and 5 are equivalent to the classical definitions in case of crisp sets and crisp structuring elements.*

Finally, the SKIZ is symmetrical with respect to the μ_i, hence independent of their order.

2.5 Illustrative example

The notion of fuzzy SKIZ is illustrated on the three objects of Figure 1. The structuring element ν is a crisp 3×3 square in Figure 2 and a fuzzy set of paraboloid shape in Figure 3. The influence zones of each object are displayed, as well as the SKIZ. These results are obtained with the dilation based definition. Each influence zone is characterized by high membership values close to the corresponding object, and decreasing when the distance to this object increases. The use of a fuzzy structuring element results in more fuzziness in the influence zones and SKIZ.

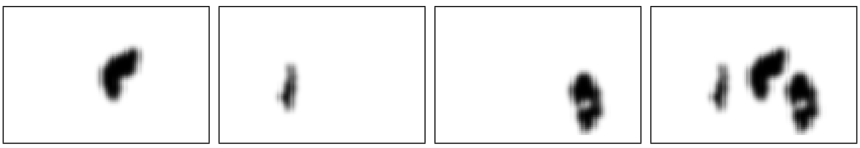

Figure 1. Three fuzzy objects and their union. Membership degrees range from 0 (white) to 1 (black).

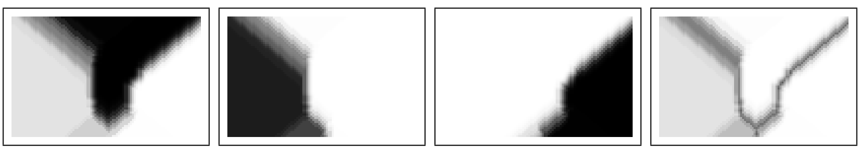

Figure 2. Influence zones of the three fuzzy objects of Figure 1 and resulting fuzzy SKIZ, obtained using a binary structuring element (3×3 square).

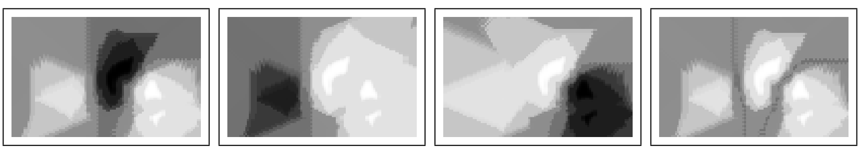

Figure 3. Influence zones of the three fuzzy objects of Figure 1 and resulting fuzzy SKIZ, obtained using a fuzzy structuring element (paraboloid shaped).

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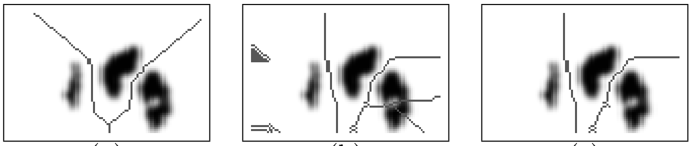

(a) (b) (c)

Figure 4. Binary decision using watershed for ν crisp (a) and fuzzy (b). Lines with a very low membership degree in the SKIZ of (b) have been suppressed in (c).

A binary decision can be made in order to obtain a crisp SKIZ of fuzzy objects. An appropriate approach consists in computing the watershed lines of the fuzzy SKIZ. It is appropriate in the sense that it provides spatially consistent lines, without holes, and going through the crest lines of the membership function of the SKIZ. A result is provided in Figure 4. For a fuzzy structuring element ν, the lines can go through the objects (Figure 4(b)). While this is impossible in the binary case, in the fuzzy case this is explained by the fact that an object can, to some degree, be built of several connected components, linked together by points with low membership degrees. The values of the SKIZ at those points are low too. This is the case for the third object in Figure 1. The low values of the SKIZ along the line traversing this object are in accordance with the fact that the object has only one connected component with some degree, and two components with some degree. The line separating the third object can be suppressed by eliminating the parts of the watersheds having a very low degree in the fuzzy SKIZ (Figure 4(c)). This requires to set a threshold value.

3. Fuzzy median set and interpolation between fuzzy sets

In the mathematical morphology community, two types of approaches have been considered to define the median set of two crisp sets, or to interpolate between two sets. The first one relies on the SKIZ [1, 19], while the second one relies on the notion of geodesics of some distance [11, 16, 17]. Here, we propose to extend the first approach to the case of fuzzy sets, based on the definitions of the fuzzy SKIZ proposed in Section 2. The median set of two intersecting sets X and Y is defined as the influence zone of $X_1 = X \cap Y$ with respect to $X_2 = (X \cup Y)^c$.

3.1 Definitions

Let us consider two fuzzy objects with membership functions μ_1 and μ_2 and with intersecting supports. Two definitions can be given for the fuzzy median set, depending on the chosen definition for the influence zones.

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Definition 6. Based on the definition of influence zones from dilations, or equivalently the direct approach from distances, the median fuzzy set of μ_1 and μ_2 is defined as the influence zone of $\mu_1 \cap \mu_2$ with respect to $(\mu_1 \cup \mu_2)^c$ (intersection is still defined by a t-norm and union by a t-conorm):

$$\begin{aligned}\forall x \in \mathcal{S}, M(\mu_1, \mu_2)(x) &= \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), 1 - \delta_{\lambda\nu}((\mu_1 \cup \mu_2)^c)(x)] \\ &= \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)(x)]. \quad (14)\end{aligned}$$

□

Definition 7. By using the definition of influence zones based on comparison of fuzzy distances, the median set is defined as:

$$M'(\mu_1, \mu_2)(x) = \sup_{n<n'} \min[d(x, \mu_1 \cap \mu_2)(n), d(x, (\mu_1 \cup \mu_2)^c)(n')]. \quad (15)$$

□

Proposition 5. *For any two fuzzy sets μ_1 and μ_2, we always have:*

$$\forall x \in \mathcal{S}, M'(\mu_1, \mu_2)(x) \leq M(\mu_1, \mu_2)(x). \quad (16)$$

This notion of median set can be exploited to derive a series of interpolating sets between μ_1 and μ_2, by applying recursively the median computation in a dichotomic process.

Definition 8. Let μ_1 and μ_2 be two fuzzy sets. A series of interpolating sets is defined by recursive application of the median computation:

$$Interp_0 = \mu_1 \quad Interp_1 = \mu_2$$
$$Interp_{\frac{i+j}{2}} = M(Interp_i, Interp_j) \text{ for } 0 \leq i \leq 1, 0 \leq j \leq 1.$$

□

This sequence allows transforming progressively μ_1 into μ_2. These two fuzzy sets can represent spatial objects, different situations, sets of constraints or preferences, etc. For instance the sequence allows building intermediate estimates between distant observations or pieces of information.

3.2 Examples

On various examples, it can be actually observed that M' leads to lower membership values than M (Proposition 5). The result provided by M is visually more satisfactory and is moreover faster to compute. Therefore, in the following the chosen definition is the one given by Equation 14. Figure 5 illustrates an example of interpolation between two fuzzy sets. The series of interpolating fuzzy sets is computed recursively from the median set (the fourth set in the sequence displayed in the figure). It is clear on this example that the shape of interpolating sets evolves progressively from the one of the first object towards the one of the second object. This evolution is in accordance with the expected interpolation notion.

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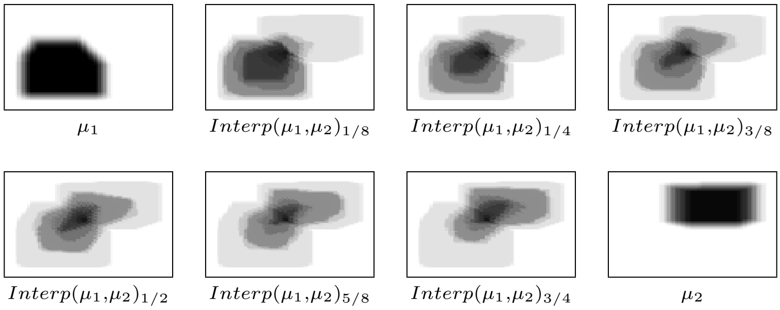

Figure 5. Interpolation between two fuzzy sets μ_1 and μ_2 ($Interp(\mu_1, \mu_2)_{1/2} = M(\mu_1, \mu_2)$).

Let us now consider real objects, from medical images. We consider the putamen (a brain structure) in different subjects, obtained from the IBSR database[1]. The images are registered, which guarantees a good correspondence between the different instances. Fuzziness at the boundary of the objects is introduced to represent spatial imprecision due to partial volume effect or imprecise segmentation, using a fuzzy dilation. Four examples of the resulting fuzzy objects are illustrated in Figure 6. The fuzzy median set has been computed between the two first instances, then between this result and the third instance, etc. Results are displayed in Figure 6. Using this iterative approach, the fuzzy median set between the 18 instances of this structure has been computed (corresponding to the 18 normal subjects of the IBSR database). Such results could be used for instance for representing the inter-individual variability, or to build anatomical atlases.

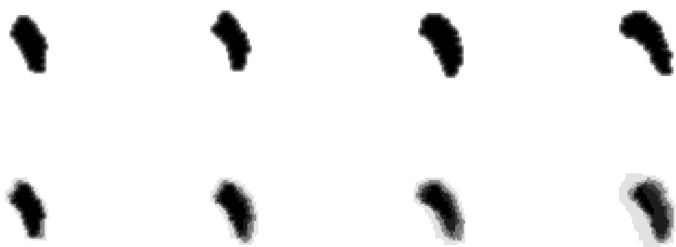

Figure 6. Four instances of a brain structure from four difference subjects, and median set between two, three, four instances and between the 18 instances of the IBSR database.

Let us now consider another example, in the domain of preference mod-

[1] http://www.cma.mgh.harvard.edu/ibsr/

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eling, as in [13], on which morphological operators can be defined [4]. We consider a propositional language based on a finite set of propositional symbols, on which formulas are defined. We denote by Ω the set of all interpretations. The models of a formula are considered as a fuzzy subset of Ω. To illustrate the application of the median operator, we consider a simple example, with three propositional symbols a, b, c, and two formulas φ_1 and φ_2, expressing respectively preferences for $\neg ab\neg c$ with a degree 0.2, and preferences for anything except abc with degrees as given in Table 1. For defining the morphological operators, we use the Hamming distance (i.e. two models are at a distance equal to the number of symbols instantiated differently), and the structuring elements are the balls of this distance. The conjunction of φ_1 and φ_2 is equal to φ_1 and their disjunction is equal to φ_2.

Table 1. Fuzzy sets of Ω representing the preferences expressed by φ_1 and φ_2, and derivation of $M(\varphi_1, \varphi_2)$.

Models	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
φ_1	0	0	0	0	0	0.2	0	0
φ_2	0	0.5	0.5	0.5	0.5	0.8	0.5	0.7
$\delta_1(\varphi_1)$	0	0.2	0	0.2	0	0.2	0	0
$\varepsilon_1(\varphi_2)$	0	0	0	0	0.5	0.5	0.5	0.5
$\delta_2(\varphi_1)$	0.2	0.2	0	0.2	0.2	0.2	0.2	0.2
$\varepsilon_2(\varphi_2)$	0	0	0	0	0	0	0	0.5
$M(\varphi_1, \varphi_2)$	0	0	0	0	0	0.2	0	0.2

The successive steps of the computation of the median set are illustrated in Table 1. The models of the median set also constitute a fuzzy set of Ω. On this example, the classical fusion, according to [12] would lead to the intersection of the sets of models, i.e. φ_1. The result of the median is somewhat larger, since it includes also a model of φ_2 that was not a model of φ_1 ($\neg a\neg b\neg c$), and gives a more fair point of view expressing an intermediate solution between both sets of preferences. This can be interpreted as follows: if an individual as a set of preferences described by φ_1, which is very strict and constraining, he will be tempted to extend his preferences to obtain a better agreement with the preferences of the second individual. On the other hand, the second individual is ready to restrict his choices to achieve a consensus with the first one, and will be more satisfied if a fair account of all his preferences is obtained. Note that the fact that the median is included in the disjunction (see Proposition 6) guarantees that it does not contain a solution that nobody wants to accept. The resulting membership degrees reflect the low consistency that exists between both sets of preferences on this example.

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3.3 Some properties

Proposition 6. *If* $\nu(0) = 1$ *(or if* ν *represents a reflexive relation), then the median set is included in the union of the two objects:* $\forall x \in \mathcal{S}, M(\mu_1, \mu_2)(x) \leq (\mu_1 \cup \mu_2)(x)$.

Proposition 7. *Under the same condition* ($\nu(0) = 1$)*, the cores verify the following inclusion relations:* $Core(\mu_1 \cap \mu_2) \subseteq Core(M(\mu_1, \mu_2)) \subseteq Core(\mu_1 \cup \mu_2)$.

The core of a fuzzy set is the set of points having a membership value equal to 1. Note that the core of the median set can be empty.

Proposition 8. *If additionally the origin is the only modal value of* ν ($\nu(0) = 1$ *and* $\forall x \in \mathcal{S} \setminus \{0\}, \nu(x) < 1$)*, then the median set and the union of the two sets have the same support and the cores of the median set and of the intersection are equal.*

In particular, in the case where the structuring element is crisp (for instance a square of size 3×3), this property does not hold, while it holds for the paraboloid shaped structuring element used in the presented results.

Figure 5 illustrates that the median set and the union have the same support. It should be noticed that in a large part of the support, the membership values are very low (0.1 in this example), and that it would be very easy to eliminate these low values if a more reduced support is desired, as could be intuitively preferable.

4. Conclusion

In this paper, novel notions of fuzzy SKIZ, median and interpolation were introduced, based on mathematical morphology concepts. The proposed definitions are applicable whatever the dimension of the underlying space $\mathcal{S}$ and whatever the semantics attached to the fuzzy sets. The only hypothesis is that it should be possible to define a structuring element, either from a distance on $\mathcal{S}$, or from a binary symmetrical relation.

The definitions of median set and interpolation can be extended to non intersecting fuzzy sets if a translation on $\mathcal{S}$ can be defined. The cases where $\mathcal{S}$ does not have an affine structure are planned for future work.

Another approach for defining median sets in the crisp case is based on geodesic distances [16]. Extension of this approach to the fuzzy case could be another interesting research direction. Extensions to a logical framework for mediation applications was proposed in [6], but not to the fuzzy sets framework until now.

Extensions of the median set to more than two fuzzy sets could be interesting too, for instance for deriving generic models from different instances. In the brain structure example, the median has been computed iteratively, a process which depends on the order. A direct method involving all objects simultaneously deserves to be developed.

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Finally, applications of the propositions of this paper could be further explored, for instance for fusion, with a comparison to other operators also based on distance [12,13], for the definition of compromises or negotiations, for smoothing fuzzy sets representing preferences, observations, etc., or for finding the fuzzy sets in between two sets.

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