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Shape parameters estimating the symmetry with respect to a point

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1. Introduction

This paper presents seven different symmetry parameters measuring the degree of symmetry with respect to a point. Some of them also determine this particular point. But in every case, those parameters enable to achieve a classification of 2D or 3D shapes according to their symmetry degree. The last three parameters are original parameters based on the determination of the similarity degree of the shapes under study to a particular shape. On the one hand, we consider the symmetrical shape according to an arbitrary point, and, on the other hand, the Minkowski symmetrical shape. The efficiency of parameters is estimated from their discrimination power. Experimental results are obtained from the processing of the presented algorithms. Let us consider S_O, the symmetry function of center O.

$$\begin{aligned} S_0 : \quad & \mathbb{R}^2 \to \mathbb{R}^2 \\ & A \mapsto A' \end{aligned}$$

And for each point A of the plane $\mathbb{R}^2$, O is the middle of the segment $[AA']$. In this way, if a shape X has a center of symmetry O_X, that means that each point of X is the image of another point of X by the central symmetry of center O_X. This definition can easily be extended to sets of $\mathbb{R}^3$.

2. Symmetry parameters

In [2], Grünbaum recalls several parameters of symmetry and gives their properties in a space of an arbitrary dimension. In the following, μ is the surface area measure in 2D or the volume measure in 3D, K is a convex body of $\mathbb{R}^2$ or $\mathbb{R}^3$, $\mathbb{K}$ is the set of convex bodies of $\mathbb{R}^2$ or $\mathbb{R}^3$ and x is a point of K. For each of the four following parameters, the convex body K is symmetric if and only if the parameter value is equal to 1. Furthermore, the lowest value is reached if K is a triangle.

Table 1 and Table 2 summarize features of four parameters described in [2] and studied in [1]. The highest value is reached for centrally symmetrical sets and the lowest for triangles.

Table 1. Definition of the symmetry parameters.

Parameter	*Function* $f(K,x)$	*Parameter value*
Besicovitch	$\frac{\mu(K \cap K^-(x))}{\mu(K)}$	$\sup_{x \in K} f(K,x)$
Blaschke	None	$\frac{\mu(K)}{\mu(S(K))}$
Minkowski	$\inf_{\theta \in [0,2\pi[} \omega(K,x,\theta)$	$\sup_{x \in K} f(K,x)$
Winternitz	$\inf_{\theta \in [0,2\pi[} \left(\frac{\mu(w_r(K,x,\theta))}{\mu(w_l(K,x,\theta))} \right)$	$\sup_{x \in K} f(K,x)$

In Table 1, $K^-(x)$ is the symmetrical set of K according to x, $K^- = K^-(O)$, $S(K) = \frac{1}{2}(K \oplus K^-)$ the Minkowski symmetrical set of K, $\omega(K,x,\theta) = \frac{\rho(K,x,\theta)}{\rho(K,x,\theta+\pi)}$ with ρ the radial function estimating the distance between x and the boundary of K in the direction θ. Figure 1 describes the partition of K used for the Winternitz function [4].

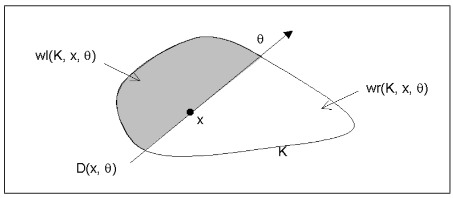

Figure 1. Partition of the convex body K according to the straight line $D(x,\theta)$.

Table 2. Symmetry parameter features.

Parameter	*Value interval*	*Center point*
Besicovitch	$[2/3, 1]$	Yes
Blaschke	$[2/3, 1]$	No
Minkowski	$[1/2, 1]$	Yes
Winternitz	$[4/5, 1]$	Yes

3. Similarity parameters estimating the symmetry degree

In order to estimate the similarity degree between shapes, let us consider a reference shape A and a shape under study X. Both of them are convex bodies. The similarity degree to A of X, $SD_A(X)$, is evaluated by the following formula [3]:

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$$SD_A(X) = \frac{1}{k_X(A_X)} \times \frac{\mu(X)}{\mu(A_X)}$$

where A_X is the smallest homothetic of A containing X and $k_X(A_X)$ the scale ratio to apply to X in order to obtain the smallest homothetic of X containing A_X.
In that way, the similarity parameter SD_A can be used as a circularity parameter if A is a disk, but also as a symmetry parameter by choosing an appropriate reference shape A. Let us compare two different symmetric shapes associated with a given convex body X. This enables to process a customized similarity degree between X and a particular symmetric shape depending on X, such as the Minkowski symmetric set for SP_1, on the one hand, or the symmetric set of X, X^- for SP_2 and SP_3, on the other hand. In both cases, the more similar the two shapes, the more X is symmetrical.
Contrary to the previous parameters (Besicovitch, Minkowski, Winternitz) the three following ones do not allow to compute a map value at each point of the convex set. Nevertheless they provide the best centered point in terms of central symmetry.
Table 3 and Table 4 summarize features of three parameters based on the determination of circumscribed convex sets. The highest value is reached for centrally symmetrical sets and the value reached for triangles is also given. Until now it has not been proved if this value is the lowest one.

Table 3. Definition of symmetry parameters based on the determination of circumscribed convex sets.

Parameter	*Parameter value*
SP_1	$SD_{X^-}(X)$
SP_2	$SD_{S(X)}(X)$
SP_3	$1/(k_{X^-}(X))$

Table 4. Values of symmetry parameters based on the determination of circumscribed convex sets.

Parameter	*Symmetrical sets*	*Triangles*
SP_1	1	1/16
SP_2	1	2/3
SP_3	1	1/2

4. Experimental results

Experimental results are given in Table 5 for three shapes: an ellipse, a triangle and a non regular hexagon.

Table 5. Values of symmetry parameters on a family of convex sets (B: Besicovitch, α: Blaschke, M: Minkowski, W: Winternitz, SP_1: similarity to the symmetrical set, SP_2: similarity to the Minkowski symmetrical set, SP_3: alternative using the symmetrical set).

Parameter	*Ellipse*	*Triangle*	*Hexagon*
B	0.9907	0.6675	0.9369
α	1.0073	0.6998	0.9580
M	0.9859	0.6047	0.9231
W	0.9410	0.7397	0.9508
SP_1	0.9723	0.0599	0.8875
SP_2	0.9155	0.6328	0.9215
SP_3	0.9901	0.4722	0.9301

5. Conclusion

All of the seven previous symmetry parameters give a quantitative estimation of the symmetry degree for convex bodies, so that these sets can be arranged in order from the less symmetrical to the most symmetrical. That is of a great interest while studying a family of convex sets. The three new parameters have still to be studied in a theoretical way in order to establish the lowest bound and the kind of sets for which it is reached. Nevertheless the experimental results show that the intuitive order from the less symmetrical shape to the most one is respected for six out of the seven studied parameters. A complete statistical study must be set up on a large set of convex bodies in order to validate these results and algorithms have to be improved to reduce bias due to estimation of geometrical features and to take into account more directions for Winternitz and Minkowski.

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