

Digital Steiner sets and Matheron semi-groups

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Abstract The Euclidean hierarchies of openings satisfy Matheron semi-groups law $\gamma_\lambda \gamma_\mu = \gamma_{\max(\lambda, \mu)}$, where λ is a size factor. One finds this law when the γ_λ are adjunction openings by Steiner convex sets, i.e. by Minkowski sums of segments. The conditions under which, in Z^n , the law remains valid, and the Steiner sets are convex, and connected, are established.

Keywords: Matheron semi-group, granulometry, digital, convexity, Steiner, connection, connectivity.

1. Matheron semi-groups and convexity

In the practice of morphological image processing, one often uses families of mappings that depend on a positive factor λ which expresses a size. When the mappings are idempotent, a convenient model for these hierarchies is the *semi-group* introduced by G. Matheron [6] Ch. 7, as a *Euclidean granulometry*, where the γ_λ 's are openings such that

$$\gamma_\lambda \gamma_\mu = \gamma_{\max(\lambda, \mu)} \quad \lambda \geq \mu \geq 0. \quad (1)$$

As a matter of fact, this law is associated with many other idempotent operators, such as the alternating sequential filters [13], or with the levelings [14]. Now all these filters derive from the two basic types of the opening by adjunction and the connected opening.

We propose to study here the discrete version of Matheron semi-groups (1) for the openings by adjunction. It is known that in the Euclidean case, these operators lie on *convex* structuring elements [6], which are the more often obtained by Minkowski sum of segments in different directions. They are then called Steiner compact sets (see Definition 1), and coincide in R^2 with the compact convex sets with a center of symmetry. Now in Z^n the Minkowski sum of two segments may be not convex, and symmetrical convex sets may not be Steiner (Figure 4(a)). Must we renounce discrete granulometries by adjunction, or deduce that convexity plays no role in

digital Matheron semi-groups? Here is the first question we have to clear up.

But it is not the only one. In vector spaces, compact convex sets are equivalently defined by barycentres, or by intersection of half-spaces. This is no longer true in Z^n (Figure 4(a)). But what can mean “a straight line segment”, or “a half-space”, in Z^n ? Must we choose among several definitions? Are they definitions for which both convexities are the same? If so, do they lead to nice hierarchies? On the other hand, what about connectivity? In R^n , but not in Z^n , convex sets are always connected (Figure 4(a)). Is it a handicap?

In digital geometry, one may either consider the module Z^n as a part of the vector space R^n , or the latter as a possible generalization of module Z^n . In our case, since we deal with structuring elements, i.e., with (necessarily discrete) actions on the objects and not with the objects themselves, the Z^n framework turns out to be the convenient one.

2. Reminders

Symbol \mathcal{L} indicates a complete lattice, whose elements are denoted by capital letters. When \mathcal{L} is of $\mathcal{P}(E)$ type, the elements of E are given by lower case letters. Let \mathcal{L} admit a class \mathcal{S} of sup-generators, and let $\{\delta_\lambda\}, \lambda \geq 0$ be a family of dilations on \mathcal{L} , of adjoint erosions $\{\varepsilon_\lambda\}$. It is known [13, 20] that the family of the openings by adjunction $\{\gamma_\lambda = \delta_\lambda \varepsilon_\lambda\}, \lambda \geq 0$, then forms a granulometry if and only if we have for all $b \in \mathcal{S}$ that

$$\lambda \geq \mu \Rightarrow \delta_\lambda(b) = \gamma_\mu \delta_\lambda(b). \quad (2)$$

In case of $\mathcal{P}(R^n)$, the families of homothetic convex sets are essential [6], because

1. the homothetic version λB of the compact set B is open by μB for all $\lambda \geq \mu$ if and only if B is convex;
2. a family $\{B_\lambda, \lambda \geq 0\}$, forms a continuous additive semi-group if and only if B_λ is homothetic of ratio λ of the compact convex set B

$$B_\lambda \oplus B_\mu = B_{\lambda+\mu}, \lambda, \mu \geq 0 \Leftrightarrow B_\lambda = \lambda B, B_\mu = \mu B, B \text{ convex.} \quad (3)$$

The link between convexity and granulometry by adjunction becomes clear, as applying Equation 2 to Euclidean granulometries, and demanding, in addition, homothetic structuring elements, i.e., $B_\lambda = \lambda B$, yields necessarily to compact convex B 's. In other words, for a Euclidean granulometry, the convexity assumption and that of homothetics are equivalent. However, if we relax the magnification assumption, then the $\delta_\lambda(b)$ do not need to be convex, nor even connected.

Among all Euclidean convex sets, the most attractive class turns out to be the *Steiner* one ([6], sect.4.5).

Definition 1. A compact set $K \in \mathcal{K}(R^n)$ is said to be *Steiner* if there exists in \mathcal{K} a sequence $\{K_n\} \in \mathcal{K}$ with $K = \text{Lim}K_n$ and if for all $n > 0$, K_n is Minkowski sum of segments centered at a same point. The Steiner class is denoted by $\mathcal{ST}(R^n)$. \square

In R^2 , Steiner sets are nothing but the symmetrical convex compact ones (rectangle, octagon, etc.) and their limits (discs, ellipses, etc.). In R^n , all 2D faces must be symmetrical convex [15]. The datum of a Steiner set K is equivalent to that of a measure $s_K = s_K(d\alpha)$ on the unit sphere Ω , since

$$K = \oplus\{s_K(d\alpha), \alpha \in \Omega\}. \quad (4)$$

For example, if K is a rectangle, then $s_K(d\alpha)$ reduces to two Dirac measures of orthogonal directions. Equation 4 has for an obvious corollary that the directional measure *exchanges* arithmetic addition and Minkowski one, i.e., $s_{K \oplus K'} = s_K + s_{K'}$, hence

$$s_{K'} \leq s_K \Rightarrow s_{K \ominus K'} = s_K - s_{K'} \Rightarrow K \text{ is open by } K'. \quad (5)$$

Consequently, every family of Steiner sets whose directional measures increase generates a granulometry by adjunction.

3. The Z^n module

Which ones of the previous results do remain when R^n is replaced by Z^n ? The poorer structure of Z^n is that of a *module*, where (integer) translation is still defined, hence giving access to Minkowski operations. The homothetic factors can only magnify, as they must be integer. Unlike integer translation, which does not pose particular problems, linear equations become a true stumbling block. However, a classical result, due to Bezout, makes precise the existence conditions of solutions.

Proposition 1. *Let $(x_1, x_2..x_n)$ be the coordinates of point $x \in Z^n$. The so called Bezout equation $\sum_1^n a_i x_i = 1$ admits solutions in Z^n if and only if the a_i coefficients relatively prime.*

The point of coordinates $(a_1, a_2..a_n)$ is usually called *Bezout vector*. When one solution \vec{u}_0 Bezout equation $\vec{a} \vec{u} = 1$ is known, then the solutions for an arbitrary second member c are given by

$$\vec{x} = c \vec{u}_0 + k_1 \vec{w}_1 + \dots k_{n-1} \vec{w}_{n-1} \quad (6)$$

where the $\vec{w}_1, \dots \vec{w}_{n-1}$ generate the sub-module $A := \{ \vec{a} \vec{x} = 0 \}$ of dimension $n - 1$. Geometrically, Equation 6 defines a *straight line* in Z^2 , a *plane* in Z^3 , etc. This geometric structure has the advantage of well scanning Z^n . One goes from the solutions of equation $\vec{a} \vec{x} = c$ to those of $\vec{a} \vec{x} = c + 1$ by replacing \vec{x} by $\vec{x} + \vec{u}$, where \vec{u} is an arbitrary solution of Bezout

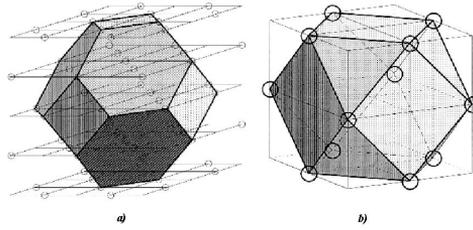


Figure 1. This is a Steiner polyhedron (a), but not that (b).

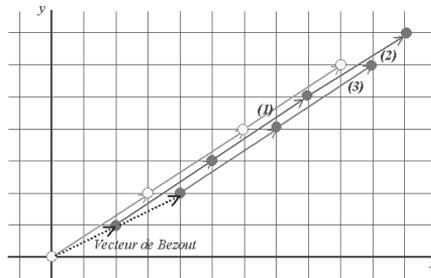


Figure 2. An example of digital plane spanning by Bezout straight lines. Vector (2,1) is a solution of Bezout equation $2x - 3y = 1$, therefore the shifts of this straight line by all multiples of vector (2,1) span integrally the plane.

equation (see Figure 2) so that, as c spans Z , every point of the space Z^n is met once and only once. This nice property is not an exclusivity of the Bezout Straight lines: H. Talbot proved in his Phd thesis that the spanning property is also satisfied by the Bresenham lines [19].

The hyperplanes of the family $\{\sum_1^n a_i x_i = c, c \in Z\}$ generate *Bezout half-spaces* $E(A, c) = \sum_1^n a_i x_i \leq c = \cup\{H(A, r), r \leq c\}$, which are nested in each other as c increases.

4. Bezout straight lines and discrete Steiner sets

Since Rosenfeld's pioneer paper [11], digital lines are the matter of an abundant literature, as well as digital convexity. The reader is referred to the survey by Eckardt [2], where at least five different ways for defining digital convexity are distinguished. Most approaches aim to provide digital representations of a Euclidean background, e.g., Rosenfeld for segments [11] or Bresenham for straight lines. Other definitions, such as Reveillès straight lines [10] are introduced in a purely digital framework. More recently, Melin proposed a digital definition in the framework of Khalimsky topology [8].

Kiselman [5] starts from Reveillès digital Equation 11, but immediately reorientates it toward the Euclidian world by making real the integers a_i .

However, the simplest, and above all the narrowest digital straight lines are given by the multiples of Bezout vectors[12, 16]. They will be our starting point.

Definition 2 (Bezout straight lines and segments). Each Bezout vector $\omega = (\omega^1, \dots, \omega^n)$ defines the *direction* ω in Z^n , the opposite direction $-\omega$ having parameters $(-\omega^1, \dots, -\omega^n)$. We call a *Bezout straight line* $D(\omega)$, of direction ω and going through the origin, the union of all integer multiples of vector ω , namely $D(\omega) = \{k\omega, k \in Z\}$. Similarly, the Bezout line of direction ω going through point x is written

$$D_x(\omega) = D(\omega) \oplus x = \{x + k\omega, k \in Z\}. \quad (7)$$

Every segment $L_x(k, \omega)$ of $D_x(\omega)$, of origin x and extremity $x + k\omega$, $k \geq 0$, consists in the sequence of the points

$$L_x(k, \omega) = x \cup \{x + p\omega, p \in [0, k]\}, \quad (8)$$

its length is the number $k + 1$ of its points. \square

In the following, the Bezout line segments are just called “segment”. The set Ω of all directions coincides with Bezout vectors, and corresponds to the unit sphere of the Euclidean case. Note also that, from Equation 7, there exists one and only one Bezout line going through a given point and with a given direction. Minkowski operations for Bezout segments are characterized by the following theorem

Theorem 1. *In $\mathcal{P}(Z^n)$, for all segments $L_x(k, \omega)$ and $L_{x'}(k', \omega)$, $\omega \in \Omega$, $x, x' \in Z^n$, $k, k' \in N$, we have that*

1. *the Minkowski sum $L_x(k, \omega)$ and $L_{x'}(k', \omega)$ is the segment*

$$L_x(k, \omega) \oplus L_{x'}(k', \omega) = L_{x+x'}(k + k', \omega), \quad (9)$$

2. *the Minkowski difference $L_x(k, \omega) \ominus L_{x'}(k', \omega)$ is $L_{x-x'}(k - k', \omega)$ if $k > k'$, $\{x - x'\}$ if $k = k'$ and $= \emptyset$ if $k < k'$,*
3. *the opening of $L_x(k, \omega)$ by $L_{x'}(k', \omega)$ is Segment $L_x(k, \omega)$ itself when $k \geq k'$ or the empty set when not.*

Conversely, the first property is only satisfied by the Bezout segments, and their periodic sub-sets, to the exclusion of the segments of any other straight line with a finite thickness.

Proof. The three properties derive from Definition 2 in the same manner. For the first one, for example, it suffices to write the Minkowski addition

$$L_x(k, \omega) \oplus L_{x'}(k', \omega) = \{x + x'\} \cup \{x + x' + (p + p')\omega, p \in [1, k], p \in [1, k']\}$$

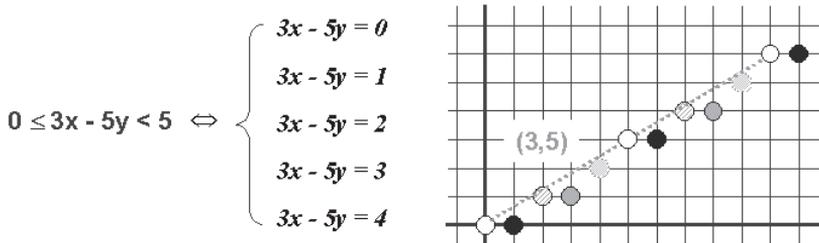


Figure 3. Decomposition of the naive Réveillès straight line $D = \{0 \leq 3x - 5y < 5\}$ into five Bezout lines.

to find Equation 9.

Conversely, suppose Equation 9 satisfied, and consider a set Δ^* that contains point x . If, as y and y' span Δ^* vector $\overrightarrow{yy'}$ always keeps the same direction, the latter can only be a multiple of a Bezout direction ω , so that y and y' describe a periodic sub-set Δ^* of the Bezout line $\Delta_x(\omega)$. If not, put the origin x in a point of Δ^* where both length direction ω and thickness direction ω' coexist. One can always find two segments $L_0(k, \omega)$ and $L_0(k', \omega')$ in Δ^* . By dilating each of them by itself, and iterating according to Equation 9, we see that Δ^* is both indefinitely long and thick. \square

In [10], Jean-Pierre Reveillès introduces a class of digital straight lines of Z^2 of variable thicknesses, by putting

$$D = \{(x, y) : t_0 \leq ax + by < t + t_0\} \quad a, b \in Z, \quad (10)$$

where a and b are relatively prime. The term t_0 corresponds to a shift that can be taken equal to zero, and the term t to the thickness of D . In particular, when $t = \max(|a|, |b|)$, line D is said to be “naive”. The extension to hyperplanes of Z^n is straightforward, one just has to replace $ax + by$ by $\sum a_i x_i$ in Equation 10. We draw from the equivalence

$$0 \leq \sum a_i x_i < t \quad \Leftrightarrow \quad \{\sum a_i x_i = s, \quad 0 \leq s < t\}, \quad (11)$$

that the hyperplanes of Equation 10 represent, for every sub-module A of dimension $n - 1$, *space slices* $\Pi(A, c, c') = E(A, c') \setminus E(A, c)$, $c' \geq c$, which are parallel to A . For example, Figure 3 depicts the decomposition of a naive Reveillès line into a union of Bezout lines.

Definition 3 (Digital Steiner set). A set $K \in \mathcal{K}(Z^n)$ is said to be *digital Steiner* if it can be decomposed into a finite Minkowski sum of Bezout segments centred at a same point. \square

Denote by $\mathcal{ST}(Z^n)$ the digital Steiner class. Theorem 1 implies that the Euclidean properties Equations 4 and (5) remain true in Z^n . Therefore every family of Steiner sets of increasing directional measures generates a granulometry by adjunction.

Clearly, the possible convexity of the Steiner sets did not play any role in the above analysis, and this point answers the first question set in introduction. We can however wonder how to link together convexity and digital Steiner sets, what we will do now.

5. Digital convexity and Steiner class

5.1 Digital convexity

The two definitions of convexity, by barycentre or by intersection of half-spaces, are no longer equivalent in Z^n as they were in R^n (see Figure 4(c)): we must choose between them. Indeed, the property of spanning of the space leads us to start from the second definition [12], p.171, [17], p.100-101.

Definition 4 (Digital Convexity). A set $X \subseteq Z^n$ is said to be *convex* when it is equal to the intersection of all Bezout half-spaces that contain it, i.e., when

$$X = \cap \{E(A, c), c \in Z, A \in \mathcal{A}\}$$

where \mathcal{A} is the set of all sub-modules of dimension $n - 1$ in Z^n . □

Given sub-module A , the smallest space slice $\Pi(A, c, c')$ parallel to A and which contains X is called the *supporting slice* $\Pi(X, A)$. We see that X is convex if and only if it is obtained by intersecting its supporting slices. This definition of the convexity implies the barycentre property since

Proposition 2. *Every Bezout segment in Z^n is convex. Moreover, if x, y are two points of a digital convex set X , then every point z of the Bezout segment $[x, y]$ belongs to X .*

Proof. We begin by the second part of the proposition. Let $\{H(\omega, c), c \in Z\}$ be a family of hyperplanes that span the space, and let c_x and c_y be the labels of the planes of two points x and y . The supporting slice $\Pi(\omega, x, y)$ generated by the hyperplanes $\{H(A, c), c \in [c_x, c_y]\}$ contains point z [15]. As set X is the intersection of its supporting half-spaces, and as for each A this intersection $C(X/A)$ contains the slice $\Pi(X, c_x, c_y)$, we can write

$$z \in \cap \{\Pi(X, c_x, c_y)\} \subseteq \cap \{C(X/A), A \in \mathcal{A}\} = X.$$

On the other hand, the intersection of all $\Pi(X, c_x, c_y)$ is nothing but the segment $[x, y]$. Indeed, suppose that a point $t \in \cap \Pi(X, c_x, c_y)$ does not belong to $[x, y]$. For any family $\{H(A, c)\}$ of hyperplanes $\{H(A, c)\}$ parallel to $[x, y]$, there exists a sense of ordering such that the labels c satisfy the conditions $c_x = c_y < c_t$, i.e., $[x, y] \in H(A, c_x)$. But $t \notin H(A, c_x)$, which implies that $t \notin \cap \Pi(X, c_x, c_y)$, i.e., that t must belong to segment $[x, y]$. □

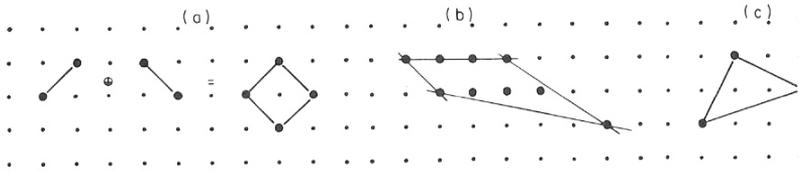


Figure 4. (a) This Steiner polygon is not convex, and, if we complete by the center of the cross, it becomes convex but is no longer Steiner; (b) convexity and connectivity are two independent notions; (c) for each pair (x, y) among the trois points, segment $[x, y]$ lies in the set, though the latter is not convex.

5.2 Convexity of the digital Steiner sets

Unlike in the Euclidean case, in Z^n the Minkowski sum of two segments of different directions is not necessarily convex (see Figure 4(a)). As the Steiner sets are built by means of such sums, the question arises to find under which conditions a directional measure $\{k_i \omega_i, 1 \leq i \leq p\}$ in Z^n corresponds to a *convex* Steiner set. We start from the following lemma:

Lemma 1. *The Minkowski dilate of Reveillès hyper-plane $\Pi = \{0 \leq \sum a_i x_i < t\}$ by Bezout vector $u = \{u_i\}$ is itself a Reveillès hyper-plane if and only if $|\sum a_i u_i| \leq t + 1$. Then it has for equation one of the two forms*

$$\Pi_u \cup \Pi = \{0 \leq \sum a_i x_i < \sum a_i u_i + t\} \text{ or } \{\sum a_i u_i \leq \sum a_i x_i < t\}. \quad (12)$$

Proof. The translate Π_u of Π has for equation $0 \leq \sum a_i (x_i - u_i) < t$, thus it is the hyper-plane $\sum a_i u_i \leq \sum a_i x_i < t + \sum a_i u_i$. The union $\Pi_u \cup \Pi$ is still an hyper-plane if and only if the two sequences of integers $[0, t]$ and $[\sum a_i u_i, t + \sum a_i u_i]$ are consecutive, i.e., when $\sum a_i u_i \leq t + 1$ or $\sum a_i u_i + t \geq -1$, or again when $|\sum a_i u_i| \leq t + 1$, which result in Equation 12. \square

Proposition 3. *The Steiner set $X \subseteq Z^n$ of directional measure $\{k_i \omega_i, 1 \leq i \leq p\}$ is convex if and only if for all directions ω_i the dilate $\Pi_i = X \oplus D_i$, where D_i is the Bezout straight line supporting L_i , is an intersection of Reveillès hyperplanes, or again if the sequence of dilations that generates X is also a sequence of contiguous translations of D_i .*

The proof of Proposition 3 is given in [15].

Figure 5 illustrates the criterion by both an example and a counter-example. Take for L_p vector $(3, 2)$. In case (a), the translations of the line $2x - 3y = 0$ by vectors L_1 and L_2 of horizontal and vertical directions lead to the lines $2x - 3y = c$, with $c = \{-3, -1, 0, 1, 2, 4\}$, which is not a contiguous series, and also X is not convex. In case (b), segment L_3 à 45 has been added, which implies $-4 \leq c \leq 4$, and also that X becomes convex.

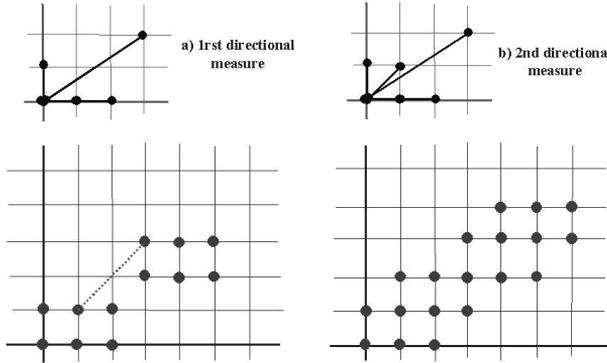


Figure 5. (a) Non convex but connected Steiner set. (b) Convex and connected Steiner set.

5.3 Connectivity of the digital Steiner sets

In Z^n , convexity does not imply connectivity (see Figure 4), even for convex Steiner sets (Figure 6(b)). However, in case of the usual arcwise connectivity, the conditions for the connectedness of a Steiner set can be found. With all point $x \in Z^n$ associate the unit cube $B(x)$ of centre x and whose points of the boundary define the extremities of the elementary arcs of origin x (e.g., the 8-connectivity in Z^2). As every cube $B(x), x \in Z^n$ contains the points which are just before and just after x in each direction of the axes, the parallelepipeds parallel to the axes are connected. Then we can state the following criterion

Proposition 4. *Let \mathcal{C} be the arcwise connection on $\mathcal{P}(Z^n)$ generated by the unit cubes $B(x), x \in Z^n$. Consider a digital Steiner set $X \subseteq Z^n$ of directional measures $\{k_i\}$ in the directions $\{\omega_i\}, 1 \leq i \leq p$, with $n \leq p$, and whose the n first directions are those of the axes of Z^n . The set X is then connected according to \mathcal{C} if and only if for each j such that $n < j \leq p$ the component ω_i^j of direction ω_j w.r.t. to axis ω_i satisfies the inequality*

$$k_j \omega_i^j \leq k_i, \quad 1 \leq i \leq n + 1, \quad n < j \leq p. \quad (13)$$

Proof. Set X is written

$$X = L_1(k_1 \omega_1) \oplus \dots \oplus L_i(k_i \omega_i) \dots \oplus L_p(k_p \omega_p) \quad (14)$$

where L_i is the vector of length k_i in direction ω_i . The dilate of origin O by the n vectors $\vec{k_i \omega_i}, 1 \leq i \leq n$, along the directions of the axes is a connected parallelepiped Π_0 . Consider one of the supplementary directions $\omega_j, n < j \leq p$. The inequality in (13) means that the extremity z_j of the segment of length k_i in direction ω_j belongs to the dilate $\Pi_0 \oplus B$. As $O \in \Pi_0$, we have that $z_j \in \Pi_{z_j}$ therefore if $z_j \in \Pi_0$ then $z_j \in \Pi_0 \cap \Pi_{z_j}$ and the union

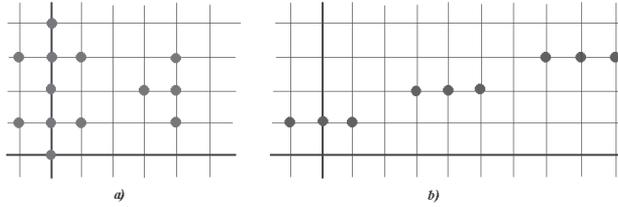


Figure 6. (a) The left set is open by the unit cross, but not by the right set, though smaller than the unit cross. (b) An example of non connected convex Steiner set.

$\Pi_0 \cup \Pi_{z_j}$ is connected. If $z_j \in (\Pi_0 \oplus B) \setminus \Pi_0$, then Π_0 and Π_{z_j} are adjacent, and their union is still connected. By iterating the proof for all directions $\omega_j, n < j \leq p$ we conclude that the Steiner set X is connected.

Conversely, suppose that X has several connected components, or “grains”. As Π_0 is connected, it is included in one of the grains X_0 de X . Then Equation 14 implies that in one of the supplementary directions at least, j say, with $n < j \leq p$, the translate of the origin by vector $k_j \omega_j$ belongs to a grain of X disjoint from X_0 , (if not, X would be a unique grain), hence disjoint from Π_0 . Therefore, for label j , the inequalities (13) are not satisfied. \square

6. Perspective vision and structuring function

There are various ways to relax digital translation invariance. The one we develop in this section aims to describe the perspective mapping. We firstly observe that in Z^1 all Steiner openings are trivial on segments (i.e., suppress them or leave them unchanged), and we want to preserve this property under perspective changes. We shall proceed by reducing Z^1 by elementary removals. Start from an opening $\gamma = \delta\varepsilon$, of extensive primitive δ , and which is trivial on segments. Remove one arbitrary point from Z^1 , taken as the origin, and and join together the two reduced half axes. The structuring elements $\{\delta(x), x \in Z^1\}$, once modified, generate a new function δ^* , still extensive and made of segments, according to the rules expressed in Figure 7, namely:

<i>Location of the extremities of</i>	$\delta(x)$	$\delta^*(x^*)$	x^*
$x_n \leq 0$	$\{x_1, \dots, x_n\}$	$\{x_1, \dots, x_n\}$	x
$x_1 \leq 0 < x_n$	$\{x_1, \dots, 0\} \{1, \dots, x_n\}$	$\{x_1, \dots, 0\} \{0, \dots, x_{n-1}\}$	0
$0 < x_1$	$\{x_1, \dots, x_n\}$	$\{x_1 - 1, \dots, x_n - 1\}$	$x - 1$

(15)

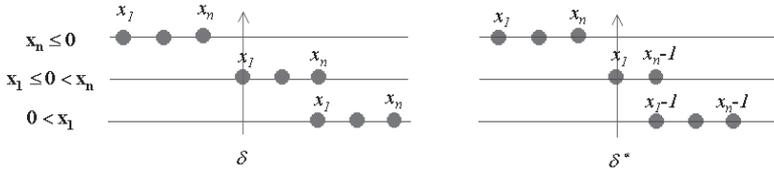


Figure 7. Bending of a structuring function δ by removal of the origin.

Consider now the action of opening $\gamma^* = \delta^* \varepsilon^*$ on a segment $L \subseteq Z^1$ of extremities y_1, y_p :

- when $y_p \leq 0$, then $\gamma^*(L) = \gamma(L)$,
- when $y_1 \leq 0 < y_p$, then, we have $\gamma^*(L) = [\gamma(L \oplus D)] \ominus D$, where D stands for doublet $\{0, 1\}$ centered at the origin,
- finally, when $0 < y_1$, then $\gamma^*(L) = \gamma(L \oplus \{1\})$.

In all cases, γ^* leaves L unchanged, or removes it, so that we can state the next proposition.

Proposition 5. *Let $\delta : Z^1 \rightarrow \mathcal{P}(Z^1)$ an extensive structuring function δ , with $\delta(x) \in \mathcal{ST}(Z^1)$, $x \in Z^1$. The structuring function δ^* defined by system (15) is in turn extensive, with $\delta^*(x) \in \mathcal{ST}(Z^1)$, $x \in Z^1$. Moreover, if the opening by adjunction $\gamma = \delta \varepsilon$ is trivial for segments, then $\gamma^* = \delta^* \varepsilon^*$ is also trivial for segments.*

The construction can be iterated, and serves when the deformations are due to an oblique perspective (e.g., T.V camera watching an a road). The space Z^2 being indicated by two axes Ox and Oy (depth), one removes all the more parallel lines to Ox since their y -ordinates increase. At the same time, in the lines which are left, the translation invariant δ are reduced as y increases.

7. Conclusion

It was shown that, in Z^n , a Steiner class can be obtained uniquely when one starts from Bezout straight lines (i.e., the narrowest digital lines), and that the resulting sets generate granulometries but may be neither convex not connected. However these two properties are reached when the Steiner class satisfies Propositions 3 and 4. Most of the results proved for Z^n extend to the sphere and to the boundary of the discrete torus. The main new results of the developed approach are given by Theorem 1, Propositions 3 and 4, and by Propositions 2 and 5.

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