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Watershed by image foresting transform, tie-zone, and theoretical relationships with other watershed definitions

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Abstract To better understand the numerous solutions related to watershed transform (WT), this paper shows the relationships between some discrete definitions of the WT: the watersheds based on image foresting transform (IFT), topographic distance (TD), local condition (LC), and minimum spanning forest (MSF). We demonstrate that the tie-zone (TZ) concept, that unifies the multiple solutions of a given WT, when applied to the IFT-WT, includes all the solutions predicted by the other paradigms: the watershed line of TD-WT is contained in the TZ of the IFT-WT, while the catchment basins of the former contain the basins of the latter; any solution of LC-WT or MSF-WT is also solution of the IFT-WT. Furthermore, the TD-WT can be seen as the TZ transform of the LC-WT.

Keywords: image segmentation, watershed transform, graph theory, minimum spanning forest, shortest-path forest.

1. Introduction

The *watershed transform* (WT) is a famous and powerful segmentation tool in morphological image processing. First introduced by Beucher and Lantuéjoul [7] for contour detection and applied in digital image segmentation by Beucher and Meyer [8], it is inspired from a physical principle well-known in geography: if a drop of water falls on a topographic surface, it follows the greatest slope until reaching a valley. The set of points which lead to the same valley is called a *(catchment) basin*. *Watershed lines* separate different basins. In the WT, an image is seen as a topographic surface where gray level corresponds to altitude. In practice, the topography is made of a gradient of the image to segment. In this case, it is expected that a region with low gradient, a valley, corresponds to a rather homogeneous region and possibly to the same object. Ideally, basins correspond to segmented objects separated by watershed lines.

Many definitions and numerous algorithms for WT exist in literature. Furthermore, multiple WT solutions are sometimes returned by an algorithm according to its implementations or even by the theoretical definition

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itself. This disconcerting fact motivated the investigation of the relationships between theoretical WT definitions.

Definitions in continuous space have been proposed [7, 18, 19, 21] and consider the watershed as a skeleton by influence zones (SKIZ) generalized to gray-scale images. In discrete space (of interest in this paper), there are many definitions which can be classified in five main paradigms. The *WT based on local condition* (LC-WT) mimics the intuitive drop of water paradigm. The inclusion of a pixel to a basin is achieved by iteratively respecting a local condition of label continuity along a path of steepest descent that reaches the basin minimum. It is why this definition includes algorithms of "arrowing", "rain simulation", "downhill", "toboggan", "hill climbing" [14, 20, 22]. The variation among them is due to processing strategy (ordered or unordered data scanning, depth- or breadth-first, union-find) and data structure.

The *WT based on flooding* has a recursive definition [23] that simulates the immersion of a topography representing the image. At each flooding level, growing catchment basins invade flooded regions that belong to their respective influence zone. The watershed corresponds to the SKIZ.

The *topological WT* [10] cannot be viewed as a generalized SKIZ but in fact, as the ultimate homotopic thinning that transforms the image while preserving some topological properties as the number of connected components of each lower cross-section and the saliency between any two (basin) minima.

The *WT based on path-cost minimization* associates a pixel to a catchment basin when the topographic distance is strictly minimum to the respective regional minimum in the case of the *WT by topographic distance* (TD-WT) [18]; or it builds a forest of minimum-path trees, each tree representing a basin, in the case of the *WT by image foresting transform* (IFT-WT) [12, 15].

The *WT based on minimum spanning forest* (MSF-WT) associates a graph to an image and builds a MSF [17], i.e., a spanning forest minimizing the sum of the weights of the arcs used for its construction. Trees correspond to basins.

Table 1 summarizes some characteristics of these WT definitions. Only flooding-WT and TD-WT definitions (not the related algorithms) return unique solution (Figure 1(b, i)), but the concept of tie zone (TZ) can be applied to the IFT-WT to unify the set of multiple solutions by creating litigious zones when solutions differ.

The LC-WT, IFT-WT (Figure 1(e–h)) and MSF-WT, are sometimes called "region"-WT because all pixels are assigned to basins, by definition. Watershed lines are considered as located between basin pixels, but can be visualized by *ad-hoc* algorithms. The other definitions are known as "line"-WT because some pixels are labeled as watershed. Yet, except for the topological WT definition (Figure 1(c, d)), they do not define lines that consistently separate basins but, instead, possibly thick and disconnected

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watershed lines.

Table 1. Characteristics of the main watershed transform (WT) definitions.

Watershed definitions	Unique solution	Watershed pixels	Separating lines	Thin lines	Grayscaled lines
LC-WT	no	no	—	—	—
Flooding-WT	yes*	yes	no	no	no
Topological-WT	no	yes	yes	no	yes
TD-WT	yes*	yes	no	no	no
IFT-WT	no	no	—	—	—
TZ-IFT-WT	yes	tie-zone	no	no	no
MSF-WT	no	no	—	—	—

* The strict definitions have a unique solution but the algorithms derived in [8, 23] do not respect the definitions and, therefore, return multiple solutions.

Observe that among these paradigms, TD-WT, IFT-WT and MSF-WT are based on a global optimality criterion. Both IFT-WT and MSF-WT are only defined in discrete space. The other paradigms attempt to mimic a continuous definition, i.e., they may be defined in both discrete and continuous spaces.

This paper shows the relationships between the discrete definitions of IFT-WT, TD-WT, MSF-WT and LC-WT. We show that the TZ watershed, derived from the solutions of the IFT-WT, contains all the solutions predicted by the other paradigms.

In Section 2, we present the IFT-WT formalism, and the TZ concept. Section 3 recalls the definition of LC-WT and demonstrates that any solution of LC-WT is also solution of the IFT-WT. Section 4 shows that the watershed region of TD-WT is contained in the TZ (derived from the IFT-WT), and the basins of the former contain the basins of the latter. In addition, the TD-WT can be seen as the TZ transform of the LC-WT. Finally, Section 5 demonstrates that any solution of MSF-WT is also solution of the IFT-WT.

2. The image foresting transform (IFT)

The IFT is a general framework based on graph theory in which an image is seen as a graph and pixels (or voxels) as its nodes. This transform returns a *shortest path forest* (SPF) from an input image-graph. Depending on the path-cost function utilized and other input parameters (adjacency, arc weights), the IFT can compute different image processing operations [11, 12]: distance transforms, connected filters, interactive object delineation ("live-wire"), segmentation by fuzzy connectedness [3] and segmentation by watershed.

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0	0	2	1	0
0	4	6	5	1
9	7	4	4	3
9	6	3	5	3
0	8	2	7	0
0	0	1	0	0

(a) Input image

A	A	**W**	B	B
A	A	**W**	B	B
A	**W**	B	B	B
W	B	B	**W**	D
C	**W**	B	**W**	D
C	C	**W**	D	D

(b) Flooding-WT

0	0	**2**	0	0
0	0	0	**2**	0
0	0	0	0	**2**
3	0	**3**	0	**3**
0	**3**	**1**	**3**	0
0	0	**1**	0	0

(c) Topological-WT 1

0	0	**2**	0	0
0	**2**	0	0	0
2	0	0	0	0
3	0	**3**	0	**3**
0	**3**	**1**	**3**	0
0	0	**1**	0	0

(d) Topological-WT 2

A	A	A	B	B
A	A	A	B	B
A	A	C	B	B
C	C	C	C	D
C	C	C	D	D
C	C	C	D	D

(e) LC-WT 1

A	A	A	B	B
A	A	A	B	B
A	D	D	B	B
C	D	D	D	D
C	C	D	D	D
C	C	D	D	D

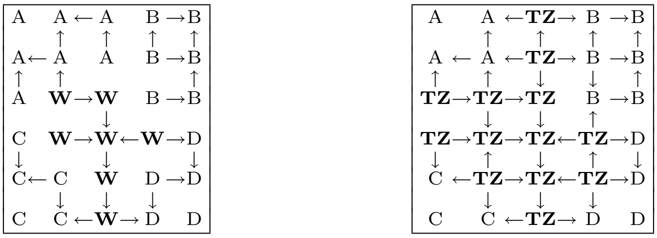
(f) LC-WT 2

A	A	A	B	B
A	A	C	B	B
C	C	C	B	B
C	C	C	C	D
C	C	C	C	D
C	C	C	D	D

(g) IFT-WT 1

A	A	B	B	B
A	A	D	B	B
D	D	D	B	B
D	D	D	D	D
C	D	D	D	D
C	C	D	D	D

(h) IFT-WT 2



(i) TD-WT and LCG   **(j)** TZ-IFT-WT and MOG

Figure 1. (a): Lower-complete input grayscale image with four minima. (b)–(j): Its WTs using 4-adjacency, according to definitions from literature. Label map is shown (**W** represents watershed line and **TZ** tie-zone) except for topological WT where watershed lines are valued. (c)–(h) show only two of the possible solutions. Watershed line in (b) and (i) is not separating. Arrows (pointing to predecessors) represent the lower complete graph (i) and multipredecessor optimal graph (j).

2.1 Watershed by image foresting transform (IFT-WT)

Under the IFT framework, an *image* is interpreted as a weighted graph $G = (V, A, w)$ consisting of a set V of *nodes* or *vertices* that represent image pixels, a set A of *arcs* weighted by w, a function from A to some nonnegative scalar domain. $N(v)$ denotes the *neighborhood* of node v, i.e., the set of nodes adjacent to it. Nodes u and v are adjacent when the arc $\langle u, v\rangle$ belongs to A. A graph (V', A') is *subgraph* of (V, A) if $V' \subseteq V$, $A' \subseteq A$ and $A' \subseteq V' \times V'$. A *forest* F of G is an acyclic subgraph F of G. *Trees* are connected components of the forest (any two nodes of a tree are connected by a path). A *path* $\pi(u, v)$ from node u to node v in graph (V, A, w) is a sequence $\langle u = v_1, v_2, \ldots, v_n = v\rangle$ of nodes of V such that $\forall i = 1 \ldots n-1$, $\langle v_i, v_{i+1}\rangle \in A$. A path is said *simple* if all its nodes are different from each other. A path with terminal node v is denoted by π_v. The path π_v is *trivial* when it consists of a single node $\langle v\rangle$. Otherwise, it can be defined by a path

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resulting from the concatenation $\pi_u \cdot \langle u, v\rangle$. A *path-cost function* f assigns to each path π a path cost $f(\pi)$, in some totally ordered set of cost values.

Let $S \subseteq V$ be a set of particular nodes s_i called *seeds*. For a given weighted graph (V, A, w) and a set S of seeds, the *image foresting transform* (IFT) returns a forest F of (V, A, w) such that (i) there exists for each node $v \in V$ a unique and simple path $\pi(s_i, v)$ in F from a seed node $s_i \in S$ to v and (ii) each such path is *optimum*, i.e., has a minimum cost for linking v to some seed of S, according to the specified path-cost function f. In other words, the IFT returns a shortest (cheapest in fact) path forest (SPF), also called *optimal forest* in this paper, where each tree is rooted to a seed. Although path costs are uniquely defined, the IFT may return many optimal forests because many paths of same minimum cost may exist for some nodes.

The *watershed transform by IFT* (IFT-WT) assumes that (i) the seeds correspond to regional minima of the image (or to imposed minima, i.e., markers [8]); (ii) the *max-arc* path-cost function $f_{\max}$ is used:

$$\begin{aligned} f_{\max}\left(\langle v\rangle\right) &= h(v), \\ f_{\max}\left(\pi_u \cdot \langle u, v\rangle\right) &= \max\left\{f_{\max}(\pi_u), w(u, v)\right\}, \end{aligned} \tag{1}$$

where $h(v)$ is a fixed but arbitrary handicap cost [16] for any paths starting at pixel v, and $w(u, v)$ is the weight of arc $\langle u, v\rangle \in A$, ideally higher on the object boundaries and lower inside the objects. Usual arc weight functions are: $w_1(u, v) = |I(u) - I(v)|$, $I(u)$ being the intensity of pixel u (cf. the so-called *watershed by dissimilarity* [15]); $w_2(u, v) = G(v)$, where $G(v)$ is the (morphological) gradient of image I at pixel v (cf. the *IFT-WT on gradient* [12, 15]). With this arc weight function, the max-arc path-cost function of Equation 1 can be simplified into: $f_{\max}(\langle v_1, v_2, \ldots, v_n\rangle) = \max\left\{G(v_1), G(v_2), \ldots, G(v_n)\right\}$. Note that the final cost map is unique and corresponds to the morphological superior reconstruction of the gradient image from the seeds using a flat structuring element. However, the forests and then the labelings may be multiple. Observe that a forest can be simply represented by a predecessor map P where $P(v)$ is the predecessor of node v in the minimum path. A label map L assigns to each node v the label $L(v)$ of the corresponding minimum-path root. The catchment basins correspond to the (labeled) trees: $CB_{IFT}(s_i) = \left\{v \in V, L(v) = L(s_i)\right\}$.

The so-called "plateau problem" is reported in WT literature for the internal non-minimum plateau pixels, i.e., non-minimum[1] pixels which have no lower neighbor. It can be solved by *lower completion* (cf. Definition 3.4 of [22]): a *lower complete image*[2] I_{LC} is computed from I by taking into account the geodesic distance of such internal pixels to the lower boundary of the plateau; then WT is applied on I_{LC}.

[1] Pixels which do not belong to regional minima.
[2] The improper term "image without plateau" is sometimes used instead.

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In IFT-WT, $f_{lex} = (f_{\max}, f_d)$, a two-component *lexicographic cost* function, is used [15] to avoid a prior lower completion but has strictly the same role [4]. The first component, of highest priority, is the max-arc function representing the flooding process. The second one corresponds to the geodesic distance to the lower boundary of the plateau and makes different waters propagate on plateau at a same speed rate:

$$f_d(\langle v_1, \ldots, v_n \rangle) = \max_{k \in [0, n-1]} \{k, f_{\max}(\langle v_1, \ldots, v_n \rangle) = f_{\max}(\langle v_1, \ldots, v_{n-k} \rangle)\} .$$

2.2 Tie zone

The choice of a single IFT-WT solution when many are possible is arbitrary and can be seen as a bias. Indeed, variations from one solution to another are sometimes significant and even unacceptable for some applications (e.g., reliable measures on segmented structures). In some images, an entire region is reached passing by a bottleneck pixel [2] and consequently included to the basin that first invades the bottleneck (like in Figure 1(g, h)). This problem is not related to the plateau problem and corresponds "to special pixel configurations which are not so rare in practice" as referred by [23].

It is why the *tie-zone concept* was proposed [4, 5] to unify the multiple solutions of a WT. Briefly speaking, considering all possible solutions derived from a specific WT definition, parts segmented in the same manner remain as catchment basins whereas differing parts are put in the tie zone (TZ). So, the TZ may be thick as well as empty.

In the case of IFT-WT, the *tie-zone watershed by IFT* (TZ-IFT-WS), returns a unique partition (cf. Figure 1(j)) of the image such that: A node is included in catchment basin $CB_{TZ-IFT}(s_i)$ when it is linked by a path to a same seed s_i in all the optimal forests (Φ denotes the set of the optimal forests F), otherwise it is included in the tie zone TZ:

$$CB_{TZ-IFT}(s_i) = \{v \in V, \quad \forall F \in \Phi, \quad \exists \pi(s_i, v) \text{ in } F\} , \qquad (2)$$
$$TZ_{IFT} = V \ \setminus \ \bigcup_i CB_{TZ-IFT}(s_i).$$

The area of the TZs, their distribution and number and distribution of their sources, the so-called bottlenecks, can be correlated with the robustness of a segmentation, i.e., with the degree of confidence a particular segmentation by WT has [2].

2.3 Multipredecessor optimal graph and lower complete graph

We introduce now a special graph, unique for each image, that will be used in Section 3. Roughly speaking, the *multipredecessor optimal graph* (MOG) of a weighted graph is the "union" of its optimal forests. More precisely, it

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is a directed acyclic subgraph of (V, A) such that its arc set A'' is the union of the (oriented) arcs of all the optimal forests $F \in \Phi$ (cf. Figure 1(j)):

$$MOG : (V, A'') = (V, \bigcup_{\forall F=(V,A')\in\Phi} A').$$

Once we have the lexicographical cost map of the image, i.e., a lower complete image, the following local property is valid: node p is predecessor of node v in the MOG if and only if p is neighbor of v with optimal lexicographic cost strictly lower than that of v (the superscript * denotes optimal paths).

$$\langle v, p\rangle \in A'' \quad \Leftrightarrow \quad p \in \mathbb{P}(v) \quad \Leftrightarrow \quad p \in N(v),\ f_{lex}(\pi^*_v) \succ f_{lex}(\pi^*_p), \tag{3}$$

where $\mathbb{P}(v)$ denotes the set of predecessors of node v, as the number of predecessors by node is no longer restricted to one as for the forests. Another property of the MOG is that if we independently choose one predecessor by non-minimum node, we obtain an optimal forest ($A' \subseteq A'' \subseteq A$).

The *lower complete graph* (V, A''') (LCG, cf. Definition 3.5 of [22]) is analog to the MOG. Both are directed acyclic graphs built from the lower complete image. While all the *lower neighbors* in the lower complete image are predecessors of a node in the MOG, only the *steepest lower neighbors* are considered for a node in LCG (cf. Figure 1(i)).

$$\langle v, p\rangle \in A''' \Leftrightarrow p \in \mathbb{P}_{steepest}(v) \Leftrightarrow \begin{array}{c} p \in N(v),\ I_{LC}(v) > I_{LC}(p), \\ \frac{I_{LC}(v)-I_{LC}(p)}{d(v,p)} = \max_{q\in N(v)} \frac{I_{LC}(v)-I_{LC}(q)}{d(v,q)}, \end{array} \tag{4}$$

$d(p, q)$ being the distance between p and q. From Equation 3 and Equation 4, we deduce that $\mathbb{P}_{steepest}(v) \subseteq \mathbb{P}(v)$. Consequently, $A''' \subseteq A'' \subseteq A$ and the LCG (V, A''') of an image-graph (V, A) is a subgraph of its MOG (V, A'').

3. Watershed based on a local condition

As we said in Section 1, the *watershed transform based on a local condition* (LC-WT) is of “region” type because it has no watershed pixels [6, 9]. It may have multiple solutions (cf. Figure 1(e, f)). It assigns to each pixel the label of some minimum m_i, so as to form a partition of the image whose disjoint sets are the basins $CB_{LC}(m_i) = \{v \in V, L(v) = L(m_i)\}$.

As observed in refs. [6, 22], this WT definition is particularly well-suited for parallel implementations because it is based on a local condition. However, the overall WT computation is still a global operation. The meaning of locality in this definition is that one may subdivide an image in blocks, do a labeling of basins in each block independently, and make the results globally consistent in a final merging step.

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Definition 1 (Watershed based on local condition)**.** For any lower complete image I_{LC}, a function L assigning a label to each pixel is called a watershed segmentation if:

1. $L(m_i) \neq L(m_j) \quad \forall i \neq j$, with $\{m_k\}$ the set of minima of I_{LC};
2. for each pixel v with $\mathbb{P}_{steepest}(v) \neq \{\}, \exists p \in \mathbb{P}_{steepest}(v), L(v) = L(p)$.

□

The condition $\mathbb{P}_{steepest}(v) \neq \{\}$ means that v has at least one lower neighbor.

In other words, we can obtain a LC-WT by independently choosing one predecessor by non-minimum node in the precomputed LCG, and assigning a different basin label to each tree of the disjoint-set forest we obtained.

As the LCG (V, A''') generating such forests is a subgraph of the MOG (V, A'') generating any optimal forest, we conclude straightaway that these forests are optimal forests. Therefore: **any LC-WT is also an IFT-WT**.

4. Watershed based on topographic distance

We recall here the definition of *WT by topographic distance* (TD-WT) and some propositions from [18] for completeness.

Definition 2 (Watershed transform by topographic distance)**.** Let I be a gray-scale image, I_{LC} its lower completion, and $\{m_i\}$ the set of minima of I. Basin of I for minimum m_i and watershed are respectively:

$$\begin{aligned} CB_{TD}(m_i) &= \{v \in V, \ \ \forall j \neq i, \ I_{LC}(m_i) + T_{I_{LC}}(v, m_i) < I_{LC}(m_j) + T_{I_{LC}}(v, m_j)\} \\ W_{TD} &= V \ \setminus \ \bigcup_i CB_{TD}(m_i) \end{aligned} \quad (5)$$

□

$T_{I_{LC}}(p, q)$ being the *topographic distance* [18] between p and q:

$$T_{I_{LC}}(p,q) = \min_{\forall \pi(p,q)} T_{I_{LC}}^{\pi(p,q)}(p,q); \ T_{I_{LC}}^{\pi(p,q)}(p = p_1, q = p_n) = \sum_{i=1}^{n-1} cost(p_i, p_{i+1});$$

$$cost(p_i, p_{i+1}) = \begin{cases} LS(p_i) d(p_i, p_{i+1}), & \text{if } I_{LC}(p_i) > I_{LC}(p_{i+1}), \\ LS(p_{i+1}) d(p_i, p_{i+1}), & \text{if } I_{LC}(p_i) < I_{LC}(p_{i+1}), \\ \frac{1}{2}\left[LS(p_i) + LS(p_{i+1})\right] d(p_i, p_{i+1}), & \text{if } I_{LC}(p_i) = I_{LC}(p_{i+1}). \end{cases}$$

The *lower slope* $LS(p)$ of I_{LC} at a pixel p is defined as the maximal slope linking p to any of its neighbors of lower altitude.

We call $(p_1, p_2, \ldots, p_n)$ a *path of steepest descent* from $p_1 = p$ to $p_n = q$ if $p_{i+1} \in \mathbb{P}_{steepest}(p_i)$ for $i = 1, \ldots, n-1$. A pixel p belongs to the *upstream* of q if there exists a path of steepest descent from p to q.

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Proposition 1 (from [18]). *Let $I_{LC}(p) > I_{LC}(q)$. A path π from p to q is of steepest descent if and only if $T^{\pi}_{I_{LC}}(p,q) = I_{LC}(p) - I_{LC}(q)$. If a path π from p to q is not of steepest descent, $T^{\pi}_{I_{LC}}(p,q) > I_{LC}(p) - I_{LC}(q)$.*

This proposition implies that paths of steepest descent are the geodesics (shortest paths) of the topographical distance function. Consequently, from Definition 2 $CB_{TD}(m_i)$ is the set of points in the upstream of a single minimum m_i, i.e., there is (at least) one path of steepest descent to m_i and no path of steepest descent to any other minimum. The watershed consists of the points p which are in the upstream of at least two minima, i.e., there are at least two paths of steepest descent starting from p which lead to different minima.

4.1 Relationship with local-condition watershed

The forests representing the possible LC-WT generated from the LCG (Section 3) are made of paths of steepest descent. By strict analogy with Equation 2, we can conclude that: **TD-WT is the tie-zone transform of LC-WT**.

Proof. A node is included in catchment basin $CB_{TD}(m_i)$ when it is linked by a path to a same minimum m_i in all the forests made of steepest paths (the set of solutions for LC-WT, e.g., Figure 1(e, f)), otherwise it is included in the tie zone W_{TD} (cf. Figure 1(i)). As a consequence, we have also $CB_{TD}(m_i) \subseteq CB_{LC}(m_i)$ (cf. Figure 1(i)), as demonstrated in Theorem 2 of [6]. □

4.2 Relationship with tie-zone watershed by IFT

We saw in Section 3 that the set of LC-WT solutions is a subset of the set of IFT-WT solutions, so **the tie zone derived from the LC-WT solutions, i.e., W_{TD}, is a subset of TZ_{IFT}: $W_{TD} \subseteq TZ_{IFT}$.**

Proof. If pixel $p \in W_{TD}$, there are at least two paths of steepest descent from p to different minima. These paths belong to the LCG and to the MOG too (LCG is subgraph of MOG). So, there exist at least two optimal forests containing these paths leading to different minima. Consequently, $p \in TZ_{IFT}$. □

Besides, **the catchment basins defined by TZ-IFT-WT are subsets of the corresponding basins defined by TD-WT:**
$\forall m_i, CB_{TZ-IFT}(m_i) \subseteq CB_{TD}(m_i)$.

Proof. If pixel $p \in CB_{TZ-IFT}(m_i)$, all the paths from p in the MOG lead to minimum m_i. So do the paths from p in the LCG (because LCG is subgraph of MOG, $\mathbb{P}_{steepest}(v) \subseteq \mathbb{P}(v), \forall v$). So, $p \in CB_{TD}(m_i)$. □

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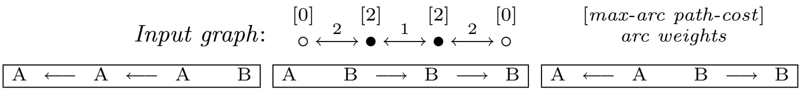

(a) MSF 1 and SPF 1 **(b)** MSF 2 and SPF 2 **(c)** SPF 3 is not a MSF

Figure 2. A weighted graph with two markers (○) and its 3 possible SPF-max and 2 MSF (total weight = 3). SPF 3 is not a MSF (total weight = 4).

5. Watershed based on a minimum spanning forest

The WT introduced in [17] is in fact a *WT from markers* (some significant minima are selected to avoid oversegmentation). It uses a weighted *neighborhood graph* whose nodes are the primitive catchment basins corresponding to regional minima of the image. Arcs are placed between neighbor catchment basins and weighted by the altitude of the *pass* between them. A *watershed based on minimum spanning forest* (MSF-WT) is defined on this weighted graph: the many possible MSFs on the graph define partitions that are considered solutions of this WT. Each tree of the MSF is a catchment basin of the MSF-WT.

A tree (V, T) is a *minimum spanning tree* (MST) of graph (V, A, w) if its total weight $\sum_{t \in T} w(t)$ (sum of the weight of its arcs) is minimum. It is unique when all the arc weights of the graph are different. A *minimum spanning forest* (MSF) is a forest whose total weight (sum of the weight of its arcs) is minimum and where each node is linked to a seed $s_i \in S$ by a unique simple path. The MSF problem for weighted graph (V, A, w) can be solved by constructing the MST of (V^*, A^*, w^*) where a ficticious root node z and arcs of weight -1 linking z to each seed were added. In a final step, these negative arcs will be removed to obtain a MSF.

Theorem 1 (Minimum spanning tree [13])**.** *(V, T) is a tree of minimum weight for graph (V, A, w) if and only if for every arc $u \in A - T$ the cycle μ^u (such that $\mu^u \subset T + \{u\}$) satisfies: $w(u) \geq w(v)$, $\forall v \in \mu^u$ $(v \neq u)$.*

Now, we demonstrate[3] that the set of MSF solutions is a subset of the set of IFT-WT solutions defined by the same weighted graph using the same seed set with seed handicaps $h(s_i) = 0$ and the max-arc path cost[4].

Theorem 2 (Shortest-path forest and minimum spanning forest)**.**
Given a weighted graph and a seed set, any minimum spanning forest (MSF) is also a shortest-path forest (SPF-max) using max-arc path cost $f_{\max}$.

$$F \textit{ is a MSF} \Rightarrow F \textit{ is a SPF-max (or IFT-WT)}.$$

[3]This result was obtained independently in [1].
[4]Until now, lexicographic path-cost $f_{lex} = (f_{\max}, f_d)$ was used for IFT-WT.

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Reciprocal is false (cf. examples and counter-example in Figure 2).

Proof. Suppose that F is a MSF and T the corresponding MST using a ficticious root z. Suppose that there exists a path π from p to z, π belongs to T and π is non optimal in the SPF-max sense (i.e., using $f_{\max}$). Suppose that there exists another path π' from p to z such that $f_{\max}(\pi') < f_{\max}(\pi)$. Then for every arc v in π', its weight $w(v) \leq f_{\max}(\pi') < f_{\max}(\pi)$. Now, there exists an arc u in π', u not in T (because T has no cycle: p and z are linked by only one simple path). Therefore, $w(u) < f_{\max}(\pi)$. Now, T is a MST. Therefore, from Theorem 1, $w(u) \geq f_{\max}(\pi \cdot \pi') \geq f_{\max}(\pi)$, $\pi \cdot \pi'$ being the cycle μ^u formed by concatenation of the two paths. That is a contradiction with the previous conclusion. So, any MSF is necessarily SPF-max. □

6. Conclusion and future works

In this paper, we used the IFT-WT and the TZ concept (that unifies the set of multiple solutions of a given WT) to relate some discrete WT definitions and, thereby, better understand the differences between the multiple solutions given by such definitions. We demonstrate that (i) the TD-WT corresponds to the tie-zone transform of the LC-WT; (ii) the possibly thick and not separating watershed line of TD-WT is contained in the TZ of the TZ-IFT-WT (with lexicographic cost function), while (iii) the catchment basins of the former contain the basins of the latter; (iv) any solution of LC-WT is also solution of the IFT-WT; (v) any solution of MSF-WT is also solution of the IFT-WT (with max-arc path-cost function).

We are preparing an extended version of this paper which will also include the comparative analysis of flooding-WT definition with IFT-WT and TZ, as well as some issues on related algorithms.

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References

[1] C. Allène, J.-Y. Audibert, M. Couprie, J. Cousty, and R. Keriven, *Some links between min cuts, optimal spanning forests and watersheds*, Procs. 8th International Symposium on Mathematical Morphology, October 2007.

[2] R. Audigier and R. A. Lotufo, *Tie-Zone Watershed, Bottlenecks and Segmentation Robustness Analysis*, XVIII Brazilian Symp. on Computer Graph. and Image Proc. (SIBGRAPI'05), October 9–12, 2005, pp. 55–62.

[3] ______, *Duality between the Watershed by Image Foresting Transform and the Fuzzy Connectedness Segmentation Approaches*, XIX Brazilian Symp. on Computer Graph. and Image Proc. (SIBGRAPI'06), October 2006, pp. 53–60.

Proceedings of the 8th International Symposium on Mathematical Morphology,
Rio de Janeiro, Brazil, Oct. 10–13, 2007, MCT/INPE, v. 1, p. 277–288.
http://urlib.net/dpi.inpe.br/ismm@80/2007/03.19.15.59

[4] ———, *Uniquely-Determined Thinning of Tie-Zone Watershed Based on Label Frequency*, Journal of Mathematical Imaging and Vision **27** (2007), no. 2, 157–173.

[5] R. Audigier, R. A. Lotufo, and M. Couprie, *The Tie-Zone Watershed: Definition, Algorithm and Applications.*, Proceedings of IEEE International Conference on Image Processing (ICIP'05), September 11–14, 2005, pp. 654–657.

[6] A. Bieniek, H. Burkhardt, H. Marschner, M. Nölle, and G. Schreiber, *A Parallel Watershed Algorithm*, Procs. 10th Scandinavian Conference on Image Analysis (SCIA'97), June 1997, pp. 237–244.

[7] S. Beucher and C. Lantuéjoul, *Use of Watersheds in Contour Detection*, International Workshop on Image Processing, Real-Time Edge and Motion Detection/Estimation, 1979.

[8] S. Beucher and F. Meyer, *The Morphological Approach to Segmentation: The Watershed Transform* (E. R. Dougherty, ed.), Marcel Dekker, Inc., New York (NY), USA, 1993, Mathematical Morphology in Image Processing, pp. 433–481.

[9] A. Bieniek and A. Moga, *A connected component approach to the watershed segmentation*, Mathematical Morphology and its Applications to Image and Signal Processing, 1998, pp. 215–222.

[10] M. Couprie and G. Bertrand, *Topological Grayscale Watershed Transformation*, SPIE Vision Geometry V Proceedings, 1997, pp. 136–146.

[11] A. X. Falcão, B. S. Cunha, and R. A. Lotufo, *Design of Connected Operators using the Image Foresting Transform*, SPIE on Medical Imaging **4322** (2001), 468–479.

[12] A. X. Falcão, J. Stolfi, and R. A. Lotufo, *The Image Foresting Transform: Theory, Algorithms, and Applications*, IEEE Transactions on Pattern Analysis and Machine Intelligence **26** (2004), no. 1, 19–29.

[13] M. Gondran and M. Minoux, *Graphs and algorithms*, John Wiley & Sons Ltd., 1984.

[14] Y. C. Lin, Y. P. Tsai, Y. P. Hung, and Z. C. Shih, *Comparison Between Immersion-Based and Toboggan-Based Watershed Image Segmentation*, IEEE Transactions on Image Processing **15** (2006), no. 3, 632–640.

[15] R. A. Lotufo and A. X. Falcão, *The Ordered Queue and the Optimality of the Watershed Approaches*, Procs. 5th International Symposium on Mathematical Morphology, June 26–28, 2000, pp. 341–350.

[16] R. A. Lotufo, A. X. Falcão, and F. A. Zampirolli, *IFT-Watershed from Gray-Scale Marker*, XV Brazilian Symp. on Computer Graph. and Image Proc. (SIBGRAPI'02), October 2002, pp. 146–152.

[17] F. Meyer, *Minimum spanning forests for morphological segmentation*, Procs. 2nd International Symposium on Mathematical Morphology, 1994, pp. 77–84.

[18] ———, *Topographic distance and watershed lines*, Signal Processing (1994), no. 38, 113–125.

[19] L. Najman and M. Schmitt, *Watershed of a continuous function*, Signal Processing **38** (1994), no. 1, 99–112.

[20] V. O. Ruiz, J. I. G. Llorente, N. S. Lechon, and P. G. Vilda, *An improved watershed algorithm based on efficient computation of shortest paths*, Pattern Recognition **40** (2007), no. 3, 1078–1090.

[21] F. Prêteux, *On a distance function approach for gray-level mathematical morphology* (E. R. Dougherty, ed.), Marcel Dekker, Inc., New York (NY), USA, 1993, Mathematical Morphology in Image Processing, pp. 323–349.

[22] J. B. T. M. Roerdink and A. Meijster, *The Watershed Transform: Definitions, Algorithms and Parallelization Strategies*, Fundamenta Informaticae **41** (2000), 187–228.

[23] L. Vincent and P. Soille, *Watersheds in digital spaces: An efficient algorithm based on immersion simulations*, IEEE Transactions on Pattern Analysis and Machine Intelligence **13** (1991), no. 6, 583–598.