

# Multi-level decomposition of Euclidean spheres

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**Abstract** Mathematical Morphology (MM) is used in medical image analysis for applications such as segmentation and skeletonization. The available efficient MM methods do not readily adapt to true Euclidean disc/sphere structuring elements (SE), which we are particularly interested in, without sacrificing accuracy for efficiency. An efficient method for MM using convex/symmetric SE, including Euclidean discs/spheres, is presented. Performance results for the proposed method are compared to the performance of a commercially available software package for a  $512 \times 512 \times 418$  chest CT dataset. Increasing gains of the method for larger SE are demonstrated, making the method suitable for analysis of high resolution images. The method is efficient for iso/anisotropic SE.

**Keywords:** structuring element decomposition, sphere, disc, circle.

## 1. Introduction

Mathematical morphology (MM) is based on set theory and can be used to analyze image shape features [5, 7, 15]. MM is useful for a wide variety of applications including object recognition, image segmentation, and industrial inspection [16]. In medical imaging, MM is used for such applications as brain segmentation from MR images [4] and airway/vessel segmentation from CT images [6, 10].

Dilation and erosion are the elementary operations of MM. Other operations such as morphological opening and closing may be formed by combining dilation and erosion in sequence [15, 16]. A structuring element (SE) is the morphological kernel that is translated over the image domain and compared with the overlapping image region. For flat (binary) SE, the comparison operation is a local maximum for dilation and a local minimum for erosion, where the local neighborhood is defined by the SE's shape. These min/max comparisons for gray-level images reduce to logical AND/OR operations for erosion/dilation of binary images using flat SE.

Morphological analysis methods used in medical imaging generally require SE of different sizes and shapes. Since some of these methods require multiple larger sized SE, which require greater processing time, such methods can take several hours for a single image [10]. Long processing times impede clinical utility of applications as well as application/algorithm development itself.

The processing time for brute-force implementation of MM is proportional to the discrete mass (number of pixels/voxels) of the SE. As such, for 2D and 3D SE, which are used for medical image processing, the processing time can increase polynomially with respect to SE diameter and can become prohibitive. A multitude of methods have been proposed to accelerate MM and the majority of these involve some form of SE decomposition. The HGW algorithm is regarded as the most efficient for a 1D straight line SE [18, 19]; logarithmic decomposition (LD) is also very efficient for 1D and as such may be applied to separable 2D and 3D SE [3, 5]. However, only a subset of useful SE are separable. Some methods are applicable only to binary images or require preprocessing/encoding of the image [9, 12]; others cater to particular constraints such as the limited region of support of specialized hardware [8, 13].

We are particularly interested in 2D Euclidean disk and 3D Euclidean sphere SE that incur no further approximation beyond quantization itself, with no restriction placed on the isotropy of the quantization. Efficient methods that have already been proposed for convex/symmetric SE [11, 21] are usually discussed and demonstrated in 2D and don't readily lend themselves to a true 3D extension. Although the sphere falls into convex/symmetric category, as far as we know, these methods may not be used to obtain the decomposition of a sphere. Moreover, the definition of convexity used in these methods is not consistent and as such, Euclidean disks/spheres are outside the scope of some such methods. Methods for decomposition of a disc SE usually sacrifice accuracy for efficiency and still do not lend themselves to the 3D sphere [1, 5, 19, 21]. Some methods refer to gray-level disk SE as spheres [1, 5]; these SE are not the focus of this paper and are not to be confused with binary 3D spheres. Although the method using local histograms described in [5] may be applied to spheres, it does not readily lend itself to an efficient vectorized implementation, which is important for certain types of computation platforms.

SE decomposition as a union of partitions is described in [2] where genetic algorithms are used to compute the decomposition. A deterministic method for decomposing a 2D disc SE into a union of partitions is introduced in [20]. However, the decomposition is described analytically and may be difficult to understand and implement. We present a reworking of this method and describe the decomposition using morphological primitives. As such, it is now easier to understand/implement. The new morphological decomposition naturally extends itself to all 2D and 3D convex symmetric SE, such as Euclidean spheres. We present timing results and comparison for a large CT lung volume dataset.

In the remainder of this paper we refer to the following notations and definitions and introduce additional elements as necessary.

$\cup$  : Union                       $\oplus$  : Dilation                       $\bullet$  : Morphological closing  
 $\cap$  : Intersection                 $\ominus$  : Erosion                       $\circ$  : Morphological opening

**Definition 1** (Convex SE). If the discrete SE is equal to the set of all voxels that fall inside its Euclidean convex hull [17] it is considered to be convex. Continuous domain convex shapes that are discretized satisfy this definition of digital convexity.  $\square$

**Definition 2** (Symmetric SE). A symmetric SE has a clearly defined center. When the center of a symmetric SE is translated to the origin of a standard Cartesian coordinate system, the translated SE would be symmetric about the  $x = 0$ ,  $y = 0$ , and  $z = 0$  planes.  $\square$

**Definition 3** (Sparse SE). In general, a 3D SE that is not face-connected, or a 2D SE that is not edge-connected, is said to be sparse.  $\square$

## 2. Method

### 2.1 Overview of the proposed decomposition

The proposed method decomposes the SE into a union of partitions (Equation 1a), where each partition,  $P_i$ , consists of two factors; the first is the largest cube,  $C_i$ , that can morphologically open the partition without change, and the second is a sparse factor  $S_i$  (Equation 1b). This method will decompose any 2D or 3D convex/symmetric SE.

$$SE = P_1 \cup P_2 \cup P_3 \dots \quad (1a)$$

$$SE = (C_1 \oplus S_1) \cup (C_2 \oplus S_2) \cup (C_3 \oplus S_3) \dots \quad (1b)$$

$$SE = (C_1 \oplus S_1) \cup ((C_1 \oplus L_2) \oplus S_2) \cup ((C_1 \oplus L_2 \oplus L_3) \oplus S_3) \dots \quad (1c)$$

$$SE = C_1 \oplus (S_1 \cup L_2 \oplus (S_2 \cup (L_3 \oplus S_3) \dots)) \quad (1d)$$

As a consequence of convexity of the underlying SE,  $C_1$  is a factor of  $C_2$  and  $C_2$  is a factor of  $C_3$  and so on (Equation 1c). The  $L_i$  factors that relate the different  $C_i$  factors are described below and illustrated in Figure 3. This allows us to further decompose  $C_i$  and reuse morphological comparisons across the cubic factors (Equation 1d).

Figure 1 illustrates Equation 1a and shows the three partitions obtained by using the proposed method to decompose the example 2D SE shown on the right hand side. The cubic and sparse factors of each of these partitions are shown in Figure 2, which illustrates Equation 1b.

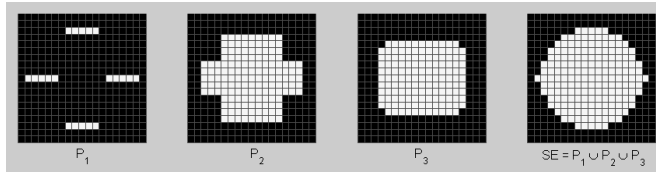


Figure 1. The proposed method decomposes the example SE (right) into a union of partitions  $P_1$ ,  $P_2$ , and  $P_3$ . The origin of each partition is the center. The partitions overlap; this is possible due to the idempotency property of comparison operations (OR, AND, min, max) used to combine the partitions.

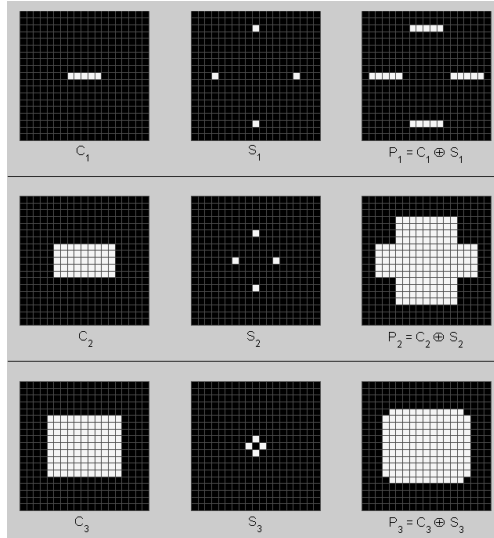


Figure 2. Each partition  $P_i$  has a cubic factor  $C_i$  and a sparse factor  $S_i$ , where  $i = 1, 2, 3$  in this example. The origin of each factor is at its center.

## 2.2 Usage of the decomposition

### Binary image dilation and erosion

$$I \oplus SE = (I \oplus P_1) \cup (I \oplus P_2) \cup (I \oplus P_3) \dots \quad (2a)$$

$$I \oplus SE = (I \oplus C_1 \oplus S_1) \cup (I \oplus C_2 \oplus S_2) \cup (I \oplus C_3 \oplus S_3) \dots \quad (2b)$$

$$I \oplus SE = I \oplus C_1 \oplus (S_1 \cup L_2 \oplus (S_2 \cup (L_3 \oplus S_3) \dots)) \quad (2c)$$

Substituting  $\oplus$  with  $\ominus$  and  $\cup$  with  $\cap$  yields the equations for erosion.

### Gray-level image dilation and erosion

$$I \oplus SE = \text{Supremum}((I \oplus P_1), (I \oplus P_2), (I \oplus P_3) \dots) \quad (3)$$

$$I \ominus SE = \text{Infimum}((I \ominus P_1), (I \ominus P_2), (I \ominus P_3) \dots) \quad (4)$$

### Efficient MM for the cubic factors $C_i$

The cubic factors of each partition may be decomposed further. Logarithmic decomposition (LD) [5] is simple and can yield efficiency through computation reuse across partitions. The logarithmic factors for the considered example are illustrated in Figure 3. The figure shows how the dilation or erosion for all three cubic factors can be obtained with merely 9 comparison operations (ops) per output pixel or voxel. This can be accomplished by using a cascaded implementation as is suggested in Equation 2c.

While we favor the cascaded implementation with LD as described above, the HGW algorithm [18] may be used to efficiently compute the morphology result for the cubic factors. However this algorithm is more complex and does not gain from computation reuse that is possible due to the fact that  $C_i$  is a factor of  $C_{i+1}$ . If the user prefers parallelization, using the proposed decomposition in the form shown in Equation 2b with the HGW algorithm might be a good choice.

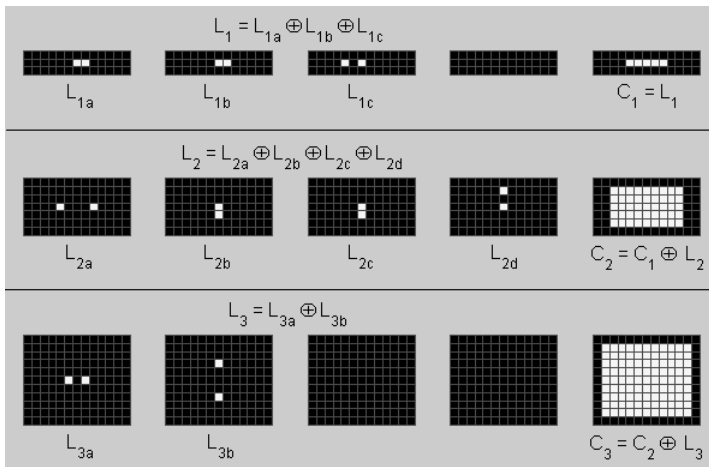


Figure 3.  $C_1$  is a factor of  $C_2$  and  $C_2$  is a factor of  $C_3$ ; as such, we may reuse computation across cubic factors. We use logarithmic decomposition (LD) to simply and efficiently perform the dilation/erosion of  $C_i$ . The origin of each  $C_i$  should be in its center. This can be satisfied either by assigning the origin of each logarithmic factor  $L_{ix}$  accordingly, or by updating the origin on each  $C_i$ . This is possible due to translational invariance property of dilation.

### 2.3 Decomposing a SE using the proposed method

Figure 4 illustrates the steps for decomposing a SE. The proposed decomposition follows the steps below.

1. Initialization: set CSE (current SE) to be equal to SE.
2. Find the largest cube with which CSE might be morphologically opened without change. This is the cubic factor  $C_i$ .

3. Find the corresponding sparse factor  $S_i$ . (See discussion below and Figure 5). Now that  $C_i$  and  $S_i$  have been found we have essentially decomposed a partition from CSE.
4. Update CSE with the subset that remains to be decomposed (RSE: remaining SE) as shown in Figure 4.
5. Go to Step 2 and repeat until RSE is NULL.

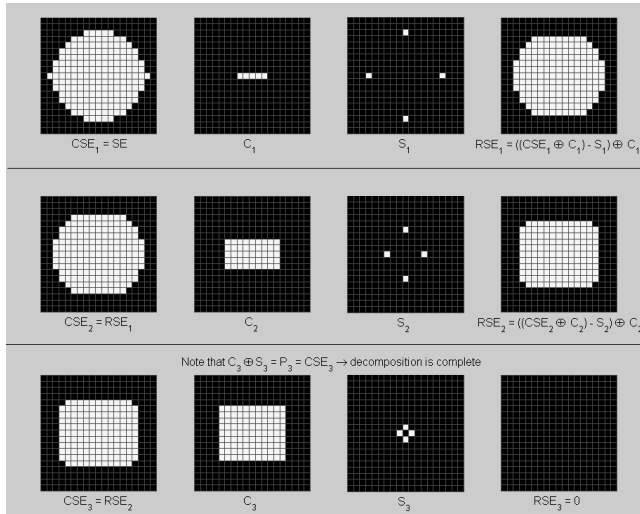


Figure 4. Decomposing the SE. At each iteration the current SE (CSE) is updated to reflect the subset of the SE that remains to be decomposed (RSE).  $C_i$  is the largest cube with which  $CSE_i$  can be morphologically opened without change. We can then determine  $S_i$  which completes the task of determining  $P_i$  and then determine the subset of  $CSE_i$  that remains to be decomposed  $RSE_i$ . The next iteration begins with updating  $CSE_{i+1}$  by setting it equal to  $RSE_i$ .

### Determining $S_i$

Figure 5 illustrates the process to determine  $S_2$  for the example in Figures 1–4.

1. Once  $C_i$  is available, obtain candidate sparse factors  $ST_1, ST_2, ST_3$ .
2. Test each  $ST_j$  for sparseness (see sub-section below).
3. Select the  $ST_j$  that is sparse and has the largest number of voxels (largest discrete mass) and assign it to  $S_i$ .

### Testing for sparseness

We enforce that our sparse factor is not face-connected for 3D. For 2D, it should not be edge-connected. There are a few exceptions to this rule, such as if there is connectivity with the central voxel (local origin) during the first iteration of decomposition, i.e., during the process of determining  $S_1$ .

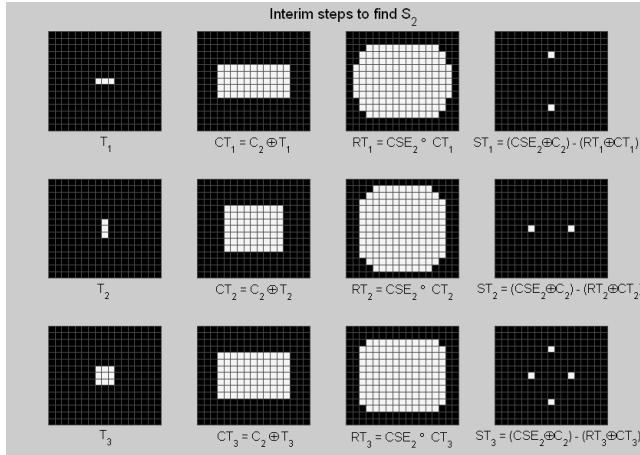


Figure 5. Determining  $S_2$  for the considered 2D example. Obtain test cubic factors  $CT_j$ , by dilating  $C_i$  with  $T_j$ , where  $j = 1, 2$ , and  $3$  for 2D SE. As illustrated above, we can then obtain a set of candidate  $RSE$ ,  $RT_j$ , and candidate sparse factors  $ST_j$ . Determine the subset of  $ST_j$  that is indeed sparse and from this subset pick the “best” one and assign it to  $S_i$ . We apply a greedy criterion for “best”, which is to pick the sparse  $ST_j$  that has the largest number of pixels/voxels. The implementation tests each  $ST_j$  for sparseness. Of course, once a particular  $ST_j$  is selected, we can simply assign its corresponding  $RT_j$  to  $RSE_i$ .

## 2.4 Method for 3D SE

- Once the decomposition is available for a 3D SE, the usage is identical to the 2D case discussed above.
- Decomposition is slightly more elaborate in the stage of selecting a sparse factor. Whereas for the 2D case we have three  $ST_j$  sparse factor candidates to consider, for the 3D case we have seven. This is a consequence of now having  $T_j$ , where  $j = 1, 2, \dots, 7$ . These  $T_j$  are the bar-3 and the square-3, each in the horizontal, vertical and axial orientations as well as the cube-3 (cube with discrete diameter 3).

### 3D sphere example

Figures 6, 7 and 8 illustrate the cascaded implementation of the proposed method for a radius-5.5 Euclidean sphere obtained under isotropic unit quantization. The method decomposes this SE into 3 partitions, as illustrated in the figures. Only the upper half of the SE is illustrated without loss of generality due to symmetry. The cubic factors  $C_1, C_2, C_3$  are equal to cube-3, cube-5, cube-9 respectively. The origin is marked in dark gray (center of slice 6 of 11) in each of the figures and is not part of  $S_1$  or  $S_2$ , which are depicted in medium gray;  $S_3$ , however, is just the single voxel at the origin. The medium and light gray voxels together depict a particular partition. The gray and white voxels together in Figure 7 illustrate the

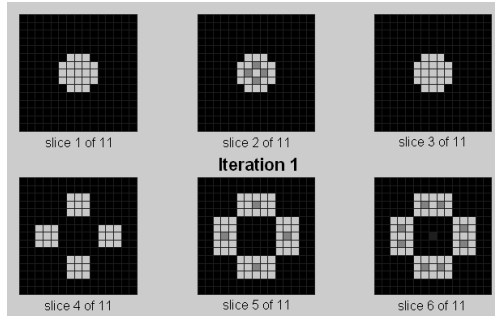


Figure 6. Upper half of  $P_1$  for a radius-5.5 Euclidean sphere at unit quantization. The origin is the dark gray voxel in the center of slice 6,  $P_1$  is the union of the light and medium gray voxels,  $S_1$  is medium gray.  $C_1$  is cube-3.

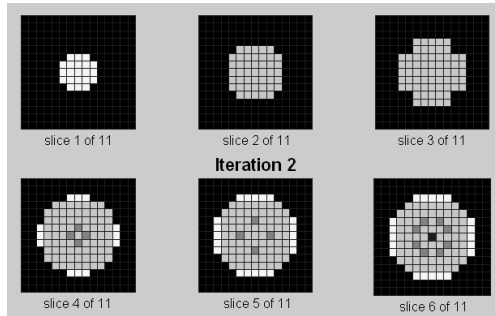


Figure 7. Foreground is  $P_1 \cup P_2$  for the radius-5.5 sphere (only upper half of the sphere is shown). The origin is dark gray,  $P_2$  is the union of light and medium gray voxels,  $S_2$  is medium gray, the region of  $P_1$  that does not overlap with  $P_2$  is white.  $C_2$  is cube-5.

union of  $P_1$  and  $P_2$ , where the white pixels indicate the portion of the SE that is exclusively covered by  $P_1$ . The union of the gray and white voxels in Figure 8 depicts the union of  $P_1$ ,  $P_2$  and  $P_3$  and is thus the SE itself. The white voxels in Figure 8 indicate the region that is covered exclusively by  $P_1 \cup P_2$ .

### 3. Results and analysis

The proposed method was implemented and compared to the commercially available SDC morphology toolbox [14]. Dilation on a gray-level chest CT image, sized  $512 \times 512 \times 418$ , was performed using various spherical SE. The standard “brute-force” direct implementation was also compared. Timing results were obtained for spherical SE with discrete radii 1-12. A dual-CPU Intel 2.0 GHz Xeon<sup>TM</sup> computer with 2 GB of RAM was used. The proposed method was implemented as a single-threaded executable. As is evident in Figure 9, the processing time for our own implementation of the



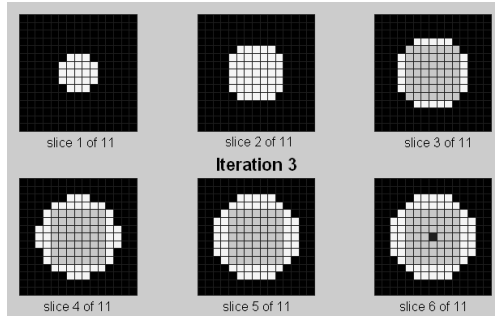


Figure 8. The origin coincides with  $S_3$  and is dark gray,  $P_3$  is the union of light gray and the origin.  $C_3$  is cube-7, which coincides with  $P_3$ . The region of  $P_1 \cup P_2$  that does not overlap with  $P_3$  is white. The union of all foreground voxels is the union of all three partitions and is the radius-5.5 Euclidean isotropic sphere.

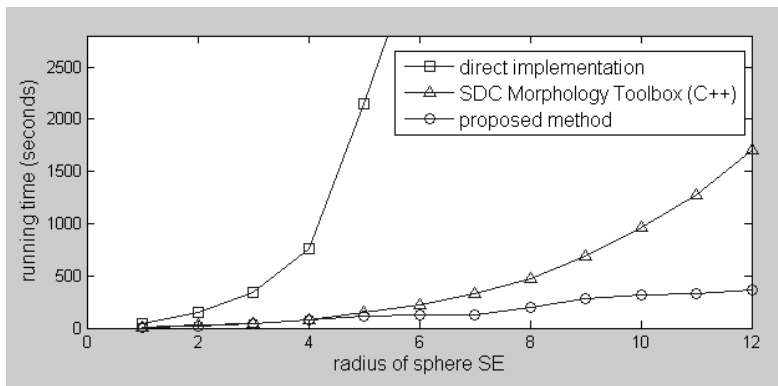
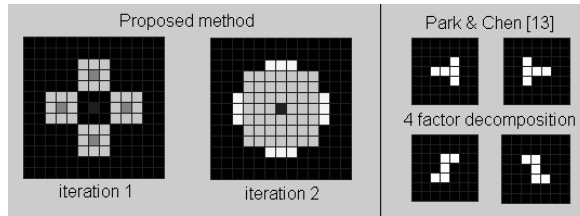


Figure 9. Processing times for gray-level dilation (binary SE) using sphere SE for a  $512 \times 512 \times 418$  chest CT image. Results obtained on and Intel Pentium IV Xeon Dual CPU 2.00 GHz platform with 2.00 GB of RAM.

direct method becomes prohibitive for larger SE. The algorithmic advantage of the proposed method facilitates gains over the SDC Morphology Toolbox that increase as SE size increases.

In [13], the four factors on the right of Figure 10 are offered as an optimal decomposition for the convex/symmetric SE shown to their left (“Iteration 2”). The proposed method will decompose the SE into 2 partitions whose cubic factors are square-3 and square-7 and obtain the MM result in 13 ops per output pixel (4 ops for  $C_1$ , 4 ops for  $L_2$ , and 5 ops for the sparse factors and the union), while the 4 factor decomposition requires 16 ops.

The number of comparison ops required per output voxel for dilation or erosion using discrete Euclidean sphere SE of radii 1-40 was calculated for the direct implementation, the proposed method, and two variations of the proposed method as shown in Figure 11. The op count for the direct implementation is one less than the number of voxels in the SE. One flavor



*Figure 10.* [Left] 2-partition decomposition of a 2D SE using the proposed method. Cubic factors are square-3 and square-7. Dark gray - origin (center): not part of  $S_1$ , however  $S_2$  is the pixel at the origin. Light and medium gray together - the partition, medium gray (only in iteration 1) -  $S_1$ , white - region of  $P_1$  that does not overlap with  $P_2$ . [Right] 4-factor decomposition for the same SE from Example 1 in [13]. The proposed method is more efficient.

of the proposed method was to consider the sphere SE as the collection of its 2D slices, implement the MM for each of the unique slices separately using the proposed method and combine the results. This implementation is less efficient and requires more memory. It illustrates that a true 3D method is superior to a 2D method directly applied to 3D SE, as would be required in order to make use of most of the applicable methods available in the literature.

The majority of comparison ops of the proposed method occur by way of the sparse factors of the partitions: for a radius-20 sphere SE, the proposed 2-canvas implementation (Equation 2c) would incur 600 comparisons per output voxel, 36 of which would be due to the cubic factors while the remaining 564 would be due to the sparse factors. By decomposing the sparse factors, this could be reduced to 292, thus reducing the total number to 328 ops per output voxel. While this could be considered an optimization, it comes at the expense of a third copy of the image volume for interim work, which may be undesirable for computation platforms with memory capacity constraints.

#### 4. Discussion and conclusion

An efficient method for MM using Euclidean disk and sphere SE has been presented. These SE do not lend themselves well to factorization methods that can usually be used for convex/symmetric SE. The proposed method can be applied to all 2D/3D convex/symmetric flat (binary) SE. The method may be used for binary as well as gray-scale images and does not require any prior analysis or encoding of the image: its gain is not based on image content. The low memory overhead and efficient looping as well as vectorized implementations promote the utility of the method for development (scripting) as well as application platforms.

We believe the challenge of decomposing SE into partitions to be an instance of an NP-complete problem. The proposed method uses locally optimal (greedy) criteria to select a sparse factor during each iteration of

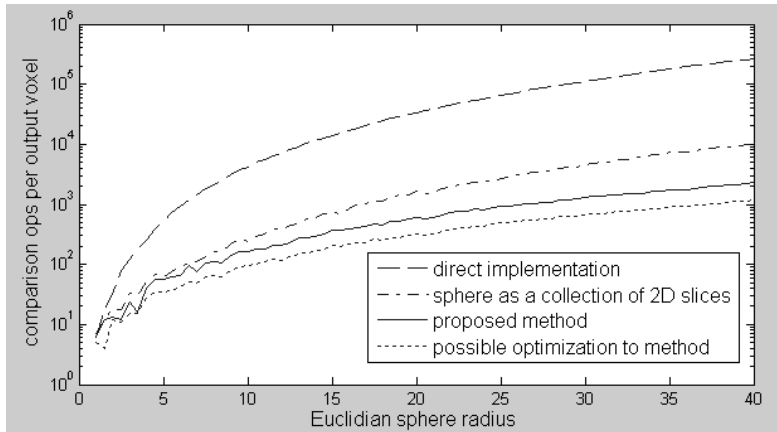


Figure 11. Number of comparison ops required per output voxel for MM using sphere SE. Note, op count presented on a log scale. Data obtained analytically. Proposed implementation follows Equation 2c and implements the sparse factors directly. A possible optimization would be to decompose the sparse factors, but this would require another interim copy of the image volume.

the decomposition. As such, it is not guaranteed to be optimal. We are investigating optimality relevant to the proposed method and expect to report on how the proposed greedy method compares to an optimal decomposition. This will of course require a clear definition of optimality.

The method is robust to scale: it works for iso/anisotropic SE. As such, it facilitates the development of scale-robust applications/algorithms. For medical image segmentation, SE may be defined in the context of anatomy instead of using discrete parameters. The chest CT image used to obtain the results shown in Figure 5 was quantized to a *mm* scale of  $0.67 \times 0.67 \times 0.80$ . With the proposed method, an algorithm could define a 3 *mm* sphere SE and this SE would be discretized according to the quantization scale of the CT image at run-time.

A comparison of the proposed method with others is underway. An empirical assessment of the gains of the method in the context of real medical applications will be performed. Since the method lends itself to vectorization, even hardware acceleration by means of a graphics processing unit (GPU) implementation would be possible.

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