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Micro-viscous morphological operators

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Abstract In most operators of mathematical morphology source and destination are the same: from pixels to pixels. In this paper we present adjunctions where source and destination are not the same. In addition to the pixels of a grid, we also consider the centers of edges linking neighboring pixels. Interesting filters may be constructed using such operators, in particular bi-levelings, where the introduction of some degree of viscosity permits to obtain higher levels of simplifications as with ordinary levelings.

Keywords: adjunction, hexagonal grid, micro-viscous operators, bi-levelings, morphological filtering.

1. Introduction

Connected filters, and in particular levelings have nice and interesting features: they simplify images without blurring the contours. For this reason they are often used as a simplification step before segmentation. Generally, one constructs a strongly simplified version g of the image f to segment, where the contours may be blurred, as is the case after Gaussian filtering, or displaced as is the case for morphological alternate sequential filters. This simplified image is called marker image. The leveling takes as input both functions f and g. It modifies g in order to restore the contours of f, by extending the regional maxima of g under f and the regional minima over f. This extension is obtained by creating flat zones. However, the result is sometimes disappointing, as the reconstruction of f reconstructs far more details as expected, if one takes into consideration the initial degree of simplification of g. For this reason one may want leveling types which reconstructs less details. Various directions have been explored. One consists in extending the regional minima and maxima of g by creating pseudo-flat zones, with a higher extension than strict flat zones [3]. Another consists in introducing some viscosity in the reconstruction process [3, 6]. This last method gives good results, but has the disadvantage to need a large support of information for processing each pixel; this slows down the processing speed and complicates the design of hardware implementations.

In the present paper, we introduce a kind of micro-viscosity in levelings which give good results without extending the window necessary for processing each pixel. It appears that the operators which are needed can be

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described in terms of adjunctions between the nodes and edges of the raster grid. They exhibit also nice filtering properties for detail simplification or for computing the gradient of noisy images.

2. Levelings and bilevelings

Reminder on levelings

Levelings [1,2] are particular connected operators: they enlarge the existing flat zones and produce new ones [4,5]. A connected operator transforms an image f into an image g in such a way that $\forall\,(p,q)$ neighbors: $g_p \neq g_q \Rightarrow f_p \neq f_q$ (0).

The relation (0) expresses that any contour between the pixels p and q in the destination image g corresponds to a contour in the initial image f at the same place. Levelings are obtained through a specialisation of Relation (0). The basic levelings are characterized by:
$\forall\,(p,q)$ neighbors: $g_p > g_q \Rightarrow f_p \geq g_p$ and $g_q \geq f_q$ (1) meaning that any transition in the destination image g is bracketed by a larger variation in the source image. Other types of transitions between neighboring pixels may be considered. For instance a minimal jump between the pixels p and q may be requested: $g_p > g_q + \lambda \Rightarrow f_p \geq g_p$ and $g_q \geq f_q$, leading to the so called λ-levelings. On the other hand g is a viscosity leveling f iff $g_p < (\gamma g)_q \Rightarrow f_p \leq g_p$ and $(\varphi g)_p < g_q \Rightarrow g_q \leq f_q$. To each type of leveling is associated a type of quasi flat-zone: two pixels x and y belong to the same quasi-flat zone of a function f, if there exists a series of pixels $\{x_0 = x, x_1, x_2, ..., x_n = y\}$ such that there is no transition between g_{x_i} and $g_{x_{i+1}}$. If transitions are of the type $g_p > g_q$, their non existence means $\{g_p \leq g_q \text{ and } g_p \geq g_q\}$, i.e., $g_p = g_q$. Similarly, the quasi-flatzones of the other leveling types are also obtained by expressing the non existence of transitions between adjacent pixels. For instance the λ-flat zones are characterized by $|g_p - g_q| < \lambda$. If g is a leveling of f, then g is identical to f except in the zones where g is quasi-flat.

The relation $g_p > g_q \Rightarrow g_q \geq f_q$ may be interpreted as $[g_p \leq g_q$ or $g_q \geq f_q]$ $\Leftrightarrow [g_q \geq f_q \wedge g_p]$. As p may be any element of the neighborhood N_q of the central point q, we obtain $g_q \geq f_q \wedge \bigvee_{x \in N_q} g_x$ (2). Since it is always true that $g_q \geq f_q \vee g_q$, Relation (2) is equivalent to $g_q \geq f_q \wedge \left(g_q \vee \bigvee_{x \in N_q} g_x \right) = f_q \wedge \delta g_q$, where δ represents the elementary morphological dilation with a flat structuring element containing the central point and all its neighbors. Taking into account the complete relation (1) yields the following criterion for the basic levelings: $f \wedge \delta g \leq g \leq f \vee \varepsilon g$ (3). Similarly $f \wedge [g \vee (\delta g - \lambda)] \leq g \leq f \vee [g \wedge (\varepsilon g + \lambda)]$ characterizes λ-levelings and $f \wedge \delta\gamma g \leq g \leq f \vee \varepsilon\varphi g$ characterizes viscous levelings (ε is the adjunct erosion of δ, γ and φ are the associated openings and closings).

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Starting with any type of simplified version g of the image f, one may transform it into a leveling of f by extending the regional maxima of g under f and the regional minima over f. This extension is obtained by creating flat zones [2]. For instance, in the case of flat levelings, one replaces g by $f \vee \varepsilon g$ for all pixels where $g > f \vee \varepsilon g$ and by $f \wedge \delta g$ for all pixels where $f \wedge \delta g > g$, until the inequalities (3) are everywhere satisfied.

Figure 2 compares the three levelings for the same noisy image; the marker is obtained by an alternate sequential filter of size 4, giving a rough approximation of structures to be preserved. It appears that the amount of simplification is higher for λ and viscous levelings than for flat levelings. In smooth regions, the λ-levelings seem to produce too large flat zones. λ-levelings seem to do a better job for filtering images before segmentation. Their drawback is to necessitate neighborhoods of size 3 for constructing the operators $\delta\gamma$ and φg. Our aim in the next section is to introduce some degree of viscosity into levelings and obtain similar results, but by using only neighborhoods of size 1; like that we accelerate the construction of viscous levelings and ease their hardware implementation.

2.1 Bilevelings

An image g is a bileveling of the image f iff $\forall\,(p,q,s)$ being the summits of an elementary triangle of the hexagonal grid:

$$g_p > g_q \text{ and } g_p > g_s \Rightarrow f_p \geq g_p, \tag{4a}$$
$$g_p < g_q \text{ and } g_p < g_s \Rightarrow f_p \leq g_p. \tag{4b}$$

A morphological characterization

Interesting characterizations may be derived from both relations. As an example consider the implication $[g_p > g_q \text{ and } g_p > g_s \Rightarrow f_p \geq g_p]$. It may be interpreted as $[g_p \leq g_q \text{ or } g_p \leq g_s \text{ or } g_p \leq f_p] \Leftrightarrow [g_p \leq f_p \vee (g_q \vee g_s)]$.

As p and s may be any couple of neighboring pixels of p, we obtain

$$g_p \leq f_p \vee \bigwedge_{(q,s,p)=triangle} (g_q \vee g_s)\,, \tag{5}$$

where $(q,s,p) = triangle$ means that (q,s,p) represent the elementary triangle of the hexagonal grid.

On the other hand, it is always true that $g \leq f \vee g$, which together with Inequality (5) is equivalent with $g_p \leq f_p \vee \left(g_p \wedge \bigwedge_{(q,s,p)=triangle} (g_q \vee g_s) \right)$.

Completing with Relation (4b), we obtain the complete characterisation of bilevelings: g is a bileveling of f iff

$$f_p \wedge \left(g_p \vee \bigvee_{(q,s,p)=triangle} (g_q \wedge g_s) \right) \leq g_p \leq f_p \vee \left(g_p \wedge \bigwedge_{(q,s,p)=triangle} (g_q \vee g_s) \right), \tag{6a}$$

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or equivalently

$$f_p \wedge \bigvee_{(q,s,p)=triangle} (g_q \wedge g_s) \leq g_p \leq f_p \vee \bigwedge_{(q,s,p)=triangle} (g_q \vee g_s). \tag{6b}$$

If g is not a bileveling of f, then the relation (6a) does not hold. So we modify g until this relation becomes satisfied:

- on $\{g_p > f_p\}$, we replace g_p by $f_p \vee \left(g_p \wedge \bigwedge_{(q,s,p)=triangle} (g_q \vee g_s) \right)$,
- on $\{g_p < f_p\}$, we replace g_p by $f_p \wedge \left(g_p \vee \bigvee_{(q,s,p)=triangle} (g_q \wedge g_s) \right)$.

In contrast to ordinary levelings, the relation $\forall (p,q)$ neighbors: $g_p > g_q \Rightarrow f_p \geq g_p$ and $g_q \geq f_q$ is not true. However if $(q,s,p) = triangle$, then $g_p > g_q > g_s \Rightarrow f_p \geq g_p$ and $g_s \geq f_s$. On the other hand, if for the same 3 pixels (q,s,p) forming a triangle we have $f_p > g_p$, $f_s > g_s$ and $f_q > g_q$,then it is not necessarily true that $g_p = g_q = g_s$, but it is granted that the two lowest values are the same.

The operators of Relations (6a) and (6b) will be reinterpret below in terms of adjunctions between the nodes and the edges of the hexagonal grid.

3. A few adjunctions on the hexagonal grid

3.1 Reminder on adjunctions

Let f be a function of Fun($\mathcal{D}$,$\mathcal{T}$) and g be a function of Fun($\mathcal{E}$,$\mathcal{T}$). The two operators α : Fun($\mathcal{D}$,$\mathcal{T}$) $\rightarrow$ Fun($\mathcal{E}$,$\mathcal{T}$) and β : Fun($\mathcal{E}$,$\mathcal{T}$) $\rightarrow$ Fun($\mathcal{D}$,$\mathcal{T}$) form an adjunction if and only if: for any f in Fun($\mathcal{D}$,$\mathcal{T}$) and g in Fun($\mathcal{E}$,$\mathcal{T}$): $\alpha f < g \Leftrightarrow f < \beta g$. Then α is a dilation (it commutes with the supremum of functions in Fun($\mathcal{D}$,$\mathcal{T}$)) and β is an erosion (it commutes with the infimum of functions in Fun($\mathcal{E}$,$\mathcal{T}$)). $\beta\alpha$ is a closing in Fun($\mathcal{D}$,$\mathcal{T}$) and $\alpha\beta$ is an opening in Fun($\mathcal{E}$,$\mathcal{T}$).

3.2 Neighborhood relations on the hexagonal grid

Let us consider a regular hexagonal grid, illustrated in Figure 1. Its basic constitutive elements are the pixels ν appearing as big (red) disks and the edges ν linking adjacent pixels (appearing in bold -green- lines). Given f, a function taking its values on any of these grids, the values taken by f on respectively ν and η are $f(\nu)$ and $f(\eta)$. As we will define a number of operators on these grids, we will consider the elements of the grid itself as operators. The operator ν applied on the function f is the value taken by f on ν : $\nu f = f(\nu)$. Similarly we define $\eta f = f(\eta)$. Let $\overline{\nu}$ be the set of nodes or pixels of the initial grid and $\overline{\eta}$ the set of edges.

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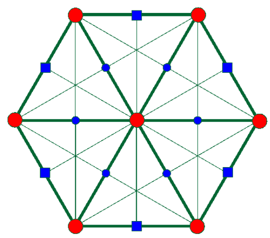

Figure 1. Hexagonal grid: the big disks are the vertices; the other dots are the centres of the vertices.

Supremum, infimum, complementation of these operators are classically defined as (we illustrate the case for $\overline{\eta}$, the definition for $\overline{\nu}$ being similar):

- $[\eta_1 \vee \eta_2](f) = \eta_1(f) \vee \eta_2(f) = f(\eta_1) \vee f(\eta_2)$;
- $[\eta_1 \wedge \vee \eta_2](f) = \eta_1(f) \wedge \eta_2(f) = f(\eta_1) \wedge f(\eta_2)$;
- $-\eta_1(f) = \eta_1(-f)$.

Neighborhood relations Each pixel ν is extremity of 6 edges; in Figure 1, the neighboring edges η of the central pixel appear as small (blue) dots; this neighboring relation is written η/ν, meaning that ν is an extremity of the edge η. Symmetrically, each edge has two extremities; this relation is written ν/η.

Each pixel ν is also summit to 6 adjacent triangles; the edges situated opposite to the central vertex ν are illustrated as (blue) squares. So each node has 6 opposing edges as neighbors; the corresponding neighborhood relation is written $\eta\backslash\nu$. Symmetrically, each edge is common to two adjacent triangles of the grid (consider in Figure 1, one of the edges marked by a small blue dot). So each edge η has as neighbors two opposing summits of triangles. This relation is written $\nu\backslash\eta$.

We now associate to each of these neighborhood relation an erosion and its dual dilation.

Relation between vertices and adjacent edges Let f be a function of Fun($\overline{\nu}$,$\mathcal{T}$) and g be a function of Fun($\overline{\eta}$,$\mathcal{T}$) . The erosion $\varepsilon_{\eta/\nu}$: Fun($\overline{\nu}$,$\mathcal{T}$) $\rightarrow$ Fun($\overline{\eta}$,$\mathcal{T}$) applied to function f is defined by its value at the edge η_i : $\eta_i \varepsilon_{\eta/\nu} f = \bigwedge_{\eta_i/\nu_j} f(\nu_j) = \bigwedge_{\eta_i/\nu_j} \nu_j f$.

Its dual operator, $\delta_{\eta/\nu}$: Fun($\overline{\nu}$,$\mathcal{T}$) $\rightarrow$ Fun($\overline{\eta}$,$\mathcal{T}$) is the dilation: $\eta_i \delta_{\eta/\nu} = \bigvee_{\eta_i/\nu_j} \nu_j$. They are indeed dual as $-\eta_i \delta_{\eta/\nu} = -\bigvee_{\eta_i/\nu_j} \nu_j = \bigwedge_{\eta_i/\nu_j} (-\nu_j)$.
Its adjunct operator maps Fun($\overline{\eta}$,$\mathcal{T}$) into Fun($\overline{\nu}$,$\mathcal{T}$) and uses the symmetrical neighborhood relation ν/η:

$$\nu_j \delta_{\nu/\eta} g = \bigvee_{\nu_j/\eta_i} g(\eta_i) = \bigvee_{\nu_j/\eta_i} \eta_i g.$$

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$\nu_j\varepsilon_{\nu/\eta}$ and $\eta_i\delta_{\eta/\nu}$ are indeed adjunct operators as
$\forall i : \eta_i\delta_{\eta/\nu} f < \eta_i g \Leftrightarrow \forall i : \bigvee_{\eta_i/\nu_j} \nu_j f < \eta_i g \Leftrightarrow$
$\forall i,j : \nu_j f < \eta_i g \Leftrightarrow \forall j : \nu_j f < \bigwedge_{\nu_j/\eta i} \eta_i g \Leftrightarrow \forall j : \nu_j f < \nu_j \varepsilon_{\nu/\eta} g.$

These four operators may be summarized in the following table, in which each row represents 2 dual operators and each column two adjunct operators:

Fun($\overline{\nu}$,$\mathcal{T}$) $\to$ Fun($\overline{\eta}$,$\mathcal{T}$)	$\eta_i\varepsilon_{\eta/\nu} = \bigwedge_{\eta_i/\nu_j} \nu_j$	$\eta_i\delta_{\eta/\nu} = \bigvee_{\eta_i/\nu_j} \nu_j$
Fun($\overline{\eta}$,$\mathcal{T}$) $\to$ Fun($\overline{\nu}$,$\mathcal{T}$)	$\nu_j\delta_{\nu/\eta} = \bigvee_{\nu_j/\eta_i} \eta_i$	$\nu_j\varepsilon_{\nu/\eta} = \bigwedge_{\nu_j/\eta_i} \eta_i$

In addition we define as previously a dilation and its dual erosion, by taking into account not only the adjacent edges for computing the value at a node but also the node itself:

Fun($\overline{\eta} \cup \overline{\nu}$,$\mathcal{T}$) $\to$ Fun($\overline{\nu}$,$\mathcal{T}$)	$\nu_j \overbrace{\delta_{\nu/\eta}} = \nu_j \vee \nu_j\delta_{\nu/\eta}$	$\nu_j \overbrace{\varepsilon_{\nu/\eta}} = \nu_j \wedge \nu_j\varepsilon_{\nu/\eta}$

The adjunct operators are defined as follows:

Fun($\overline{\nu}$,$\mathcal{T}$) $\to$ Fun($\overline{\eta} \cup \overline{\nu}$, $\mathcal{T}$)	$\nu_j \overbrace{\varepsilon_{\eta/\nu}} = \nu_j$ $\eta_i \overbrace{\varepsilon_{\eta/\nu}} = \eta_i\varepsilon_{\eta/\nu}$	$\nu_j \overbrace{\delta_{\eta/\nu}} = \nu_j$ $\eta_i \overbrace{\delta_{\eta/\nu}} = \eta_i\delta_{\eta/\nu}$

Relation between vertices and opposing edges Similar operators may be based on the neighborhood relations between the nodes and the edges which are in opposition to them, i.e., the neighborhood relations $\eta\backslash\nu$ and $\nu\backslash\eta$. These four operators may be summarized in the following table, in which each row represents 2 dual operators and each column two adjunct operators:

Fun($\overline{\nu}$,$\mathcal{T}$) $\to$ Fun($\overline{\eta}$,$\mathcal{T}$)	$\eta_i\varepsilon_{\eta\backslash\nu} = \bigwedge_{\eta_i\backslash\nu_j} \nu_j$	$\eta_i\delta_{\eta\backslash\nu} = \bigvee_{\eta_i\backslash\nu_j} \nu_j$
Fun($\overline{\eta}$,$\mathcal{T}$) $\to$ Fun($\overline{\nu}$,$\mathcal{T}$)	$\nu_j\delta_{\nu\backslash\eta} = \bigvee_{\nu_j\backslash\eta_i} \eta_i$	$\nu_j\varepsilon_{\nu\backslash\eta} = \bigwedge_{\nu_j\backslash\eta_i} \eta_i$

In addition we define as previously a dilation and its dual erosion, by taking into account not only the opposing edges for computing the value at a node but also the node itself:

Fun($\overline{\eta} \cup \overline{\nu}$,$\mathcal{T}$) $\to$ Fun($\overline{\nu}$,$\mathcal{T}$)	$\nu_j \overbrace{\delta_{\nu\backslash\eta}} = \nu_j \vee \nu_j\delta_{\nu\backslash\eta}$	$\nu_j \overbrace{\varepsilon_{\nu\backslash\eta}} = \nu_j \wedge \nu_j\varepsilon_{\nu\backslash\eta}$

The corresponding adjunct operators are defined as follows:

Fun($\overline{\nu}$,$\mathcal{T}$) $\to$ Fun($\overline{\eta} \cup \overline{\nu}$, $\mathcal{T}$)	$\nu_j \overbrace{\varepsilon_{\eta\backslash\nu}} = \nu_j$ $\eta_i \overbrace{\varepsilon_{\eta\backslash\nu}} = \eta_i\varepsilon_{\eta\backslash\nu}$	$\nu_j \overbrace{\delta_{\eta\backslash\nu}} = \nu_j$ $\eta_i \overbrace{\delta_{\eta\backslash\nu}} = \eta_i\delta_{\eta\backslash\nu}$

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3.3 Reinterpretation of the micro-viscous operators used in the bilevelings

The four operators used to characterize and to build the bilevelings can be now reinterpreted in terms of adjunctions between the nodes and the edges of the hexagonal grid:

- $\bigwedge_{(q,s,p)=triangle} (g_q \vee g_s) = \bigwedge_{\nu_p \backslash \eta_i} \eta_i \delta_{\eta/\nu} g = \nu_p \varepsilon_{\nu\backslash\eta} \delta_{\eta/\nu} g;$
- $\bigvee_{(q,s,p)=triangle} (g_q \wedge g_s) = \bigvee_{\nu_p \backslash \eta_i} \eta_i \varepsilon_{\eta/\nu} g = \nu_p \delta_{\nu\backslash\eta} \varepsilon_{\eta/\nu} g;$
- $g_p \wedge \bigwedge_{(q,s,p)=triangle} (g_q \vee g_s) = g_p \wedge \bigwedge_{\nu_p \backslash \eta_i} \eta_i \delta_{\eta/\nu} g = \nu_p g \wedge \nu_p \varepsilon_{\nu\backslash\eta} \delta_{\eta/\nu} g = \nu_p \overbrace{\varepsilon_{\nu\backslash\eta}} \delta_{\eta/\nu} g;$
- $g_p \vee \bigvee_{(q,s,p)=triangle} (g_q \wedge g_s) = g_p \vee \bigvee_{\nu_p \backslash \eta_i} \eta_i \varepsilon_{\eta/\nu} g = \nu_p g \vee \nu_p \delta_{\nu\backslash\eta} \varepsilon_{\eta/\nu} g = \nu_p \overbrace{\delta_{\nu\backslash\eta}} \varepsilon_{\eta/\nu} g.$

Figure 2 presents a detail for a noisy version of the image “Barbara” (Gaussian noise $\sigma = 20$, $SNR(dB) = 20,13$). The marker function for all the examples is an alternate sequential filter of size 4, giving a rough approximation of structures to be preserved. We compare the results of 5 levelings: the standard flat leveling, the lambda leveling ($\lambda = 1$), the viscous leveling (based on $\delta\gamma$ and $\varepsilon\varphi$), a bileveling based on the micro-viscous operators $\overbrace{\delta_{\nu\backslash\eta}} \varepsilon_{\eta/\nu}$ and $\overbrace{\varepsilon_{\nu\backslash\eta}} \delta_{\eta/\nu}$, and finally a lambda bileveling (defined by the microviscous-operators $g \wedge \big(\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu} g + \lambda\big)$ and $g \vee \big(\delta_{\nu\backslash\eta}\varepsilon_{\eta/\nu} g - \lambda\big)$). In addition, the corresponding flat zones are given for each leveled image. It is evident that viscous levelings and bilevelings lead to stronger detail simplification and enlargement of (quasi-)flat zones, especially in noisy and texture images like this example. It seems also that the micro-viscous bilevelings preserve the localisation of the original contours better than the viscous leveling. The micro-operations between vertices and edges, and edges and vertices seem interesting to introduce the viscosity in levelings despite their small size.

4. Micro-viscous pseudo-erosions and dilations, derived operators

4.1 Pseudo-erosions and dilations

For an image f of Fun($\overline{\nu}$,$\mathcal{T}$) the hexagonal unitary erosion εf : Fun($\overline{\nu}$,$\mathcal{T}$) $\rightarrow$ Fun($\overline{\nu}$,$\mathcal{T}$) (resp. dilation δf) is based on computing the infimum (resp. supremum) of 6 nodes together with the center node in the hexagonal neighborhood, i.e., $\nu_i \varepsilon f = \bigwedge_{\nu_j} f(\nu_j) = \bigwedge_{\nu_j} \nu_j f$. The unitary operation can

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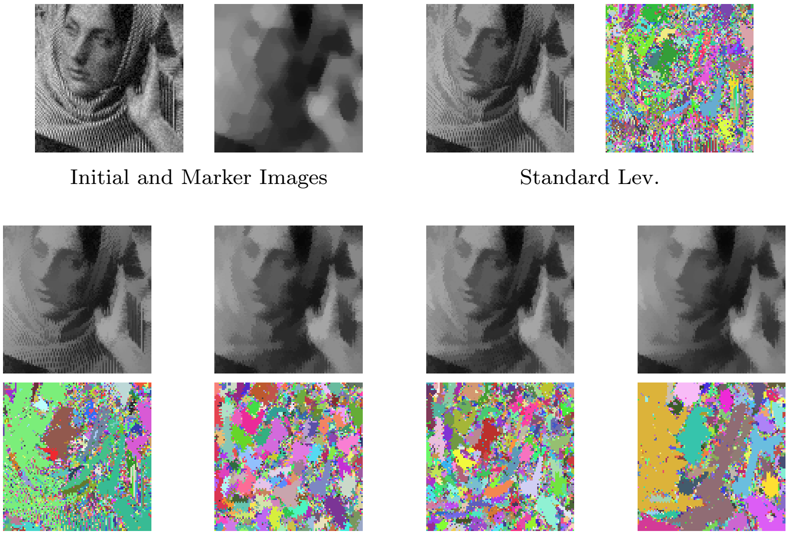

Initial and Marker Images | Standard Lev.

λ-Lev. | Viscos. Lev. | Micro-viscos. Bilev. | λ-Micro-viscos. Bilev.

Figure 2. Comparison of standard vs. viscous levelings and bilevelings and their associated λ-flat zones ($\lambda = 1$).

be iterated n-times to build an operator of size n, i.e., $[\varepsilon f]^n$. Similarly to the bilevelings, the micro-viscous operators associated to the adjunctions between the nodes and the edges can be used to introduce other morphological operators.

We propose two uncentered pseudo-erosions for the image f:

- $\xi_{\nu\backslash\eta/\nu} f = \varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu} f$,
- $\xi_{\nu/\eta\backslash\nu} f = \varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu} f$,

where $\xi_{\nu\backslash\eta/\nu}$, $\xi_{\nu/\eta\backslash\nu}$: $\mathrm{Fun}(\overline{\nu},\mathcal{T}) \to \mathrm{Fun}(\overline{\nu},\mathcal{T})$. The corresponding uncentered pseudo-dilations $\tau_{\nu\backslash\eta/\nu} f$ and $\tau_{\nu/\eta\backslash\nu} f$ are obtained by taking the dual micro-operators. We denote by $[\xi_{\nu\backslash\eta/\nu} f]^n$ the pseudo-erosion of size n obtained by iteration of n unitary operators. These operators are called pseudo-erosions (resp. pseudo-dilations) in $\mathrm{Fun}(\overline{\nu})$ because they are increasing (as product of increasing operators) but they do not commute with the infimum (resp. supremum) and the antiextensivity (resp. extensivitiy) in $\overline{\nu}$ is not guaranteed.

Using the micro-viscous operators which take into account the center node during the operation between edges and nodes, it is also possible to define two centered pseudo-erosions:

- $\overbrace{\xi_{\nu\backslash\eta/\nu}} f = \overbrace{\varepsilon_{\nu\backslash\eta}}\delta_{\eta/\nu} f$,

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- $\overbrace{\xi_{\nu/\eta\backslash\nu}} f = \overbrace{\varepsilon_{\nu/\eta}} \delta_{\eta\backslash\nu} f$,

and by duality are obtained the two respective centered pseudo-dilations: $\overbrace{\tau_{\nu\backslash\eta/\nu}} f$ and $\overbrace{\tau_{\nu/\eta\backslash\nu}} f$. The centered pseudo-erosions (resp. pseudo-dilations) have the same properties as the uncentered pseudo-erosions (resp. pseudo-dilations) but in addition they are antiextensive (resp. extensive) in $\overline{\nu}$. In addition, both operators $\overbrace{\xi_{\nu\backslash\eta/\nu}} f$ and $\overbrace{\xi_{\nu/\eta\backslash\nu}} f$ are $\geq \varepsilon f$: centered pseudo-erosions are weaker than standard erosions.

4.2 Pseudo-inverses and other evolved operators

To each operator defined above, one may associate its pseudo-inverse operator, obtained by concatenating in reverse order the adjunct operators:

- $\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu} \rightarrow \varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}$;
- $\delta_{\nu\backslash\eta}\varepsilon_{\eta/\nu} \rightarrow \delta_{\nu/\eta}\varepsilon_{\eta\backslash\nu}$;
- $\overbrace{\varepsilon_{\nu\backslash\eta}}\delta_{\eta/\nu} \rightarrow \varepsilon_{\nu/\eta}\overbrace{\delta_{\eta\backslash\nu}} = \varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}$;
- $\overbrace{\delta_{\nu\backslash\eta}}\varepsilon_{\eta/\nu} \rightarrow \delta_{\nu/\eta}\overbrace{\varepsilon_{\eta\backslash\nu}} = \delta_{\nu/\eta}\varepsilon_{\eta\backslash\nu}$.

Concatenating such an operator with its pseudo-inverse produces for instance $\varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu}$: its construction introduces the opening $\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}$ within the closing $\varepsilon_{\nu/\eta}\delta_{\eta/\nu}$.

This operator is increasing, being the product of increasing operators, but it is not a filter as it is not idempotent. However it is an underfilter:

$$\begin{aligned}\varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu}\varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu} &\leq \varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu}\\ &= \varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu},\end{aligned}$$

since $\delta_{\eta/\nu}\varepsilon_{\nu/\eta}$ is antiextensive and $\delta_{\eta\backslash\nu}\varepsilon_{\nu\backslash\eta}$ is idempotent.

Similarly $\overbrace{\varepsilon_{\nu\backslash\eta}}\delta_{\eta/\nu}\varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}$ is an underfilter whereas $\delta_{\nu\backslash\eta}\varepsilon_{\eta/\nu}\delta_{\nu/\eta}\varepsilon_{\eta\backslash\nu}$ and $\overbrace{\varepsilon_{\nu\backslash\eta}}\delta_{\eta/\nu}\varepsilon_{\nu/\eta}\delta_{\eta\backslash\nu}$ are overfilters.

Note that other operators can be defined as a product of unit pseudo-erosions/dilations. For instance, a pseudo-opening can be defined as

$$\tau_{\nu\backslash\eta/\nu}\xi_{\nu\backslash\eta/\nu} = \delta_{\nu\backslash\eta}\varepsilon_{\eta/\nu}\varepsilon_{\nu\backslash\eta}\delta_{\eta/\nu},$$

the corresponding pseudo-closing is obtained as $\xi_{\nu\backslash\eta/\nu}\tau_{\nu\backslash\eta/\nu}$, and *mutatis mutandis* other pseudo-openings and closings are obtained with the other unitary micro-operations. Then, the product of pseudo-openings and closings leads to more evolved operators such as the pseudo-alternate sequential filters (pseudo-ASF). A detailed study of the properties of these operators is out of the scope of this paper but we would like to show a few examples. In

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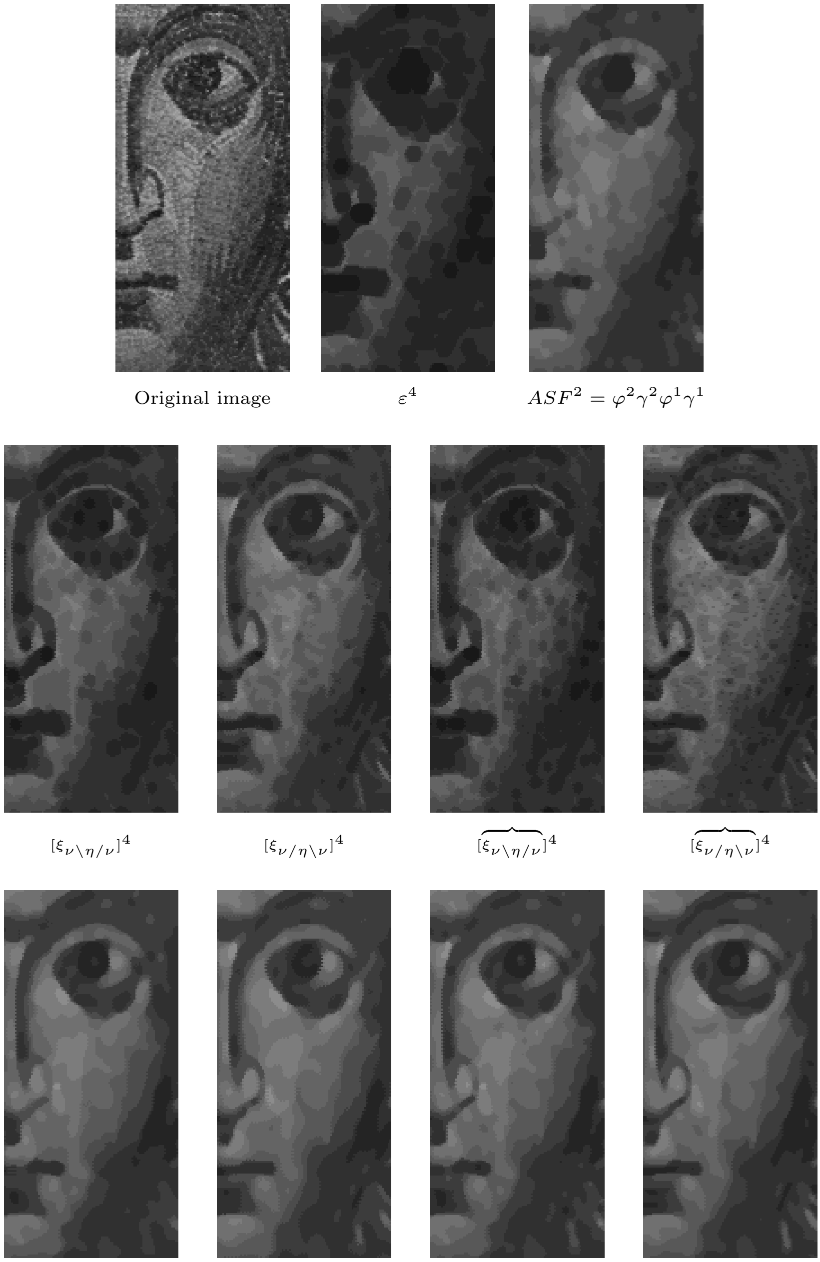

Original image | ε^4 | $ASF^2 = \varphi^2\gamma^2\varphi^1\gamma^1$

$[\xi_{\nu\backslash\eta/\nu}]^4$ | $[\xi_{\nu/\eta\backslash\nu}]^4$ | $[\overbrace{\xi_{\nu\backslash\eta/\nu}}]^4$ | $[\overbrace{\xi_{\nu/\eta\backslash\nu}}]^4$

$ASF^2_{\xi_{\nu\backslash\eta/\nu},\tau_{\nu\backslash\eta/\nu}}$ | $ASF^2_{\xi_{\nu/\eta\backslash\nu},\tau_{\nu/\eta\backslash\nu}}$ | $ASF^2_{\overbrace{\xi_{\nu\backslash\eta/\nu}},\overbrace{\tau_{\nu\backslash\eta/\nu}}}$ | $ASF^2_{\overbrace{\xi_{\nu/\eta\backslash\nu}},\overbrace{\tau_{\nu/\eta\backslash\nu}}}$

Figure 3. Comparison of micro-viscous operators: uncentered and centered pseudo-erosions of size 4 (second row) and uncentered and centered pseudo-alternate sequential filters of size 2 (third row) using a detail of the image "Greek Mosaic" (the original image, the standard hexagonal erosion of size 4 and the standard hexagonal alternate sequential filter of size 4, are given in the first row).

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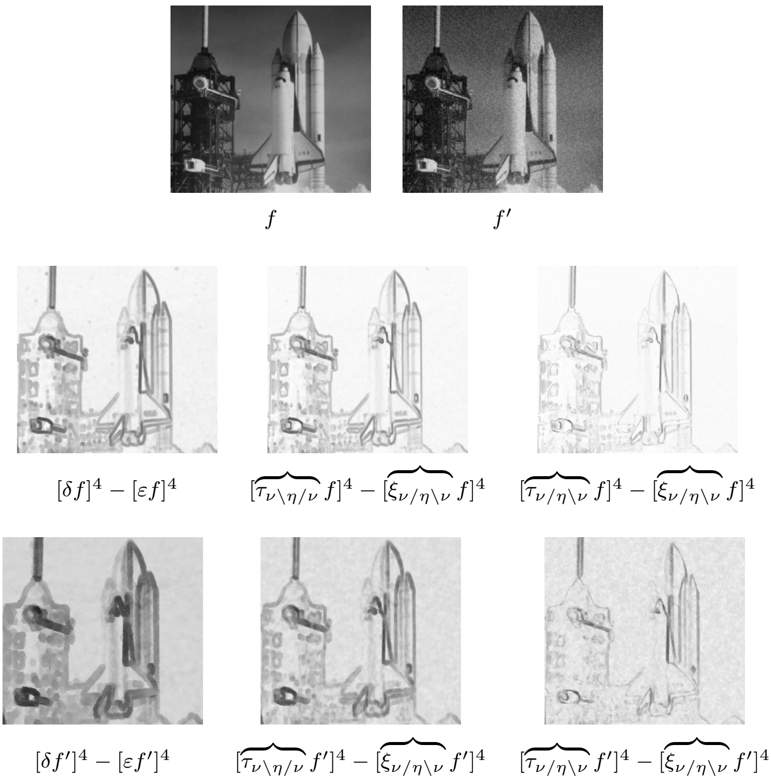
Figure 3 is given a comparison of the different micro-viscous pseudo-erosions and pseudo-alternate sequential filters presented above on a detail of image "Greek Mosaic". These results of filtering are compared with corresponding standard hexagonal operators. Both kinds of micro-viscous operators $\xi_{\nu\backslash\eta/\nu}$ and $\xi_{\nu/\eta\backslash\nu}$, and the derived pseudo-ASF, have more selective effects than the standard hexagonal counterpart operators, and they result in a regularization of contours. The effects of operators starting from opposing edges η/ν are stronger than starting from adjacent edges $\eta\backslash\nu$: this is due to the distance between the pair of nodes used to compute the edge value. Moreover, note that the uncentered pseudo-erosions (and the derived operators) which are nor antiextensive neither extensive, perform very well for detail simplification; in contrast the centered pseudo-erosions propagate the isolated dark details.



f f'

$[\delta f]^4 - [\varepsilon f]^4$ $[\overbrace{\tau_{\nu\backslash\eta/\nu}} f]^4 - [\overbrace{\xi_{\nu/\eta\backslash\nu}} f]^4$ $[\overbrace{\tau_{\nu/\eta\backslash\nu}} f]^4 - [\overbrace{\xi_{\nu/\eta\backslash\nu}} f]^4$

$[\delta f']^4 - [\varepsilon f']^4$ $[\overbrace{\tau_{\nu\backslash\eta/\nu}} f']^4 - [\overbrace{\xi_{\nu/\eta\backslash\nu}} f']^4$ $[\overbrace{\tau_{\nu/\eta\backslash\nu}} f']^4 - [\overbrace{\xi_{\nu/\eta\backslash\nu}} f']^4$

Figure 4. Comparison of standard vs. micro-viscous centered thick-gradient of size 4 (negative of gradients are shown): Top, "Shuttle" original and corrupted image (gaussian noise $\sigma = 20$, $SNR(dB) = 19,37$); middle, thick gradients for original image and down, thick gradients for noisy image.

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The properties of micro-viscous pseudo-erosions/dilations make them interesting for instance to define gradients which are robust to noise. Figure 4 depicts a comparison of standard vs. micro-viscous centered thick-gradient of size 4 for the image "Shuttle" and its corrupted version with gaussian noise. The thick-gradient is defined as the difference between the (pseudo-) dilation and the (pseudo-)erosion of size 4. As we can observe, the gradients based on micro-viscous operators are much more robust to noise than the standard ones. Moreover, the thickness of contours depends strongly on the chosen couple of operations between the nodes and the edges.

5. Conclusions and perspectives

The hexagonal grid offers the highest degree of rotational symmetry and a dense packing of pixels. The implementation of microviscous operators on this grid is simple and elegant. However, it is not complicated to implement similar operators on the square grid, and more generally on weighted graphs: it is sufficient to define erosions and dilations between nodes, adjacent edges and adjacent faces.

The extensions of this work will be in three directions: (i) explore more completely all adjunctions between sub-elements of the hexagonal and square grid, (ii) extend the method to various grids in 3D, (iii) extend the method to arbitrary weighted neighborhood graphs.

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