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On a generative topology for the digital plane

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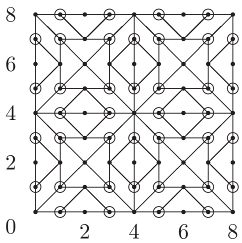
1. Introduction

We study a special topology on $\mathbb{Z}^2$ and show that both the Khalimsky and the Marcus-Wyse topologies as well as one more digital topology and a closure operator on $\mathbb{Z}^2$ may be obtained as its quotients.

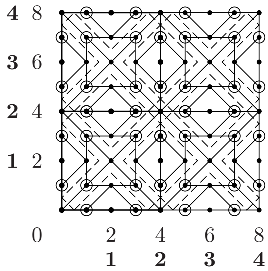
Throughout the text, topologies are thought of as being (given by) Kuratowski closure operators. Let p be a topology on $\mathbb{Z}^2$. By a *simple closed curve* in the topological space $(\mathbb{Z}^2, p)$ we mean a nonempty, finite and connected subset $C \subseteq \mathbb{Z}^2$ such that, for each point $x \in C$, there are exactly two points of C adjacent to x in the connectedness graph of p. A simple closed curve C in $(\mathbb{Z}^2, p)$ is said to be a *Jordan curve* if it separates $(\mathbb{Z}^2, p)$ into precisely two components (i.e., if the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, p)$ consists of precisely two components).

2. A generative topology on $\mathbb{Z}^2$ and some of its quotients

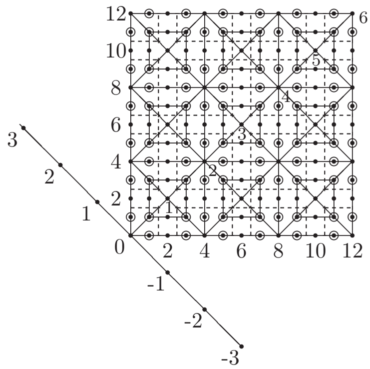
In every connectedness graph displayed in this section, the closed points are ringed. We denote by w the Alexandroff $T_{\frac{1}{2}}$-topolog on $\mathbb{Z}^2$ with a portion of its connectedness graph shown in the following figure:



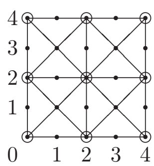
Theorem 1. *The Khalimsky plane is homeomorphic to the quotient topological space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the bold coordinates.*



Theorem 2. *The Marcus-Wyse plane is homeomorphic to the quotient topological space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the coordinates with respect to the diagonal axes.*



Let v be the Alexandroff $T_{\frac{1}{2}}$-topology on $\mathbb{Z}^2$ with a portion of its connectedness graph shown in the following figure:



Theorem 3. $(\mathbb{Z}^2, v)$ *is homeomorphic to the quotient topological space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the bold coordinates.*

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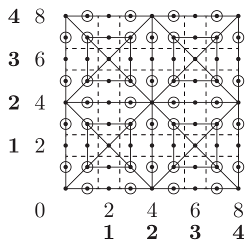
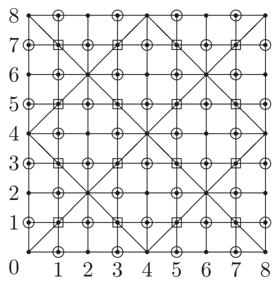
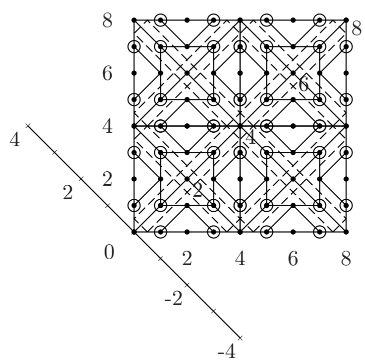


By a closure space we understand a set endowed with a Čech closure operator [1], i.e., a closure operator fulfilling all the Kuratowski closure axioms with a possible exception of idempotency. Let u be the Alexandroff T_0-closure operator on $\mathbb{Z}^2$ with a portion of its connectedness graph shown in the following figure where the points that are neither closed nor open are boxed:



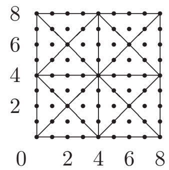
Theorem 4. *$(\mathbb{Z}^2, u)$ is homeomorphic to the quotient closure space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the coordinates with respect to the diagonal axes.*



3. Digital Jordan curve theorems

The following result is proved in [7]:

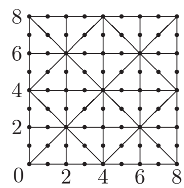
Theorem 5. *Every cycle in the graph (with the vertex set $\mathbb{Z}^2$) a portion of which is shown in the following figure is a Jordan curve in $(\mathbb{Z}^2, w)$:*



Theorems 1-5 enable us to identify Jordan curves among the simple closed curves in the Khalimsky and Marcus-Wyse planes, in $(\mathbb{Z}^2, v)$ and in $(\mathbb{Z}^2, u)$. For example, in addition to the digital Jordan curve theorems proved in [7], we get

a) *Let D be a simple closed curve in $(\mathbb{Z}^2, v)$ with the property that, for each point $(x, y) \in D$, there exists $k \in \mathbb{Z}$ such that $y = x+4k+2$ or $y = 4k+2-x$. Then D is a Jordan curve in $(\mathbb{Z}^2, v)$.*

b) *Every simple closed curve in $(\mathbb{Z}^2, u)$ that is a cycle in the graph a portion of which is shown in the following figure is a Jordan curve in $(\mathbb{Z}^2, u)$:*



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