On a generative topology for the digital plane

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1. Introduction

We study a special topology on $\mathbb{Z}^2$ and show that both the Khalimsky and the Marcus-Wyse topologies as well as one more digital topology and a closure operator on $\mathbb{Z}^2$ may be obtained as its quotients.

Throughout the text, topologies are thought of as being (given by) Kuratowski closure operators. Let $p$ be a topology on $\mathbb{Z}^2$. By a simple closed curve in the topological space $(\mathbb{Z}^2, p)$ we mean a nonempty, finite and connected subset $C \subseteq \mathbb{Z}^2$ such that, for each point $x \in C$, there are exactly two points of $C$ adjacent to $x$ in the connectedness graph of $p$. A simple closed curve $C$ in $(\mathbb{Z}^2, p)$ is said to be a Jordan curve if it separates $(\mathbb{Z}^2, p)$ into precisely two components (i.e., if the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, p)$ consists of precisely two components).

2. A generative topology on $\mathbb{Z}^2$ and some of its quotients

In every connectedness graph displayed in this section, the closed points are ringed. We denote by $w$ the Alexandroff $T_4$-topology on $\mathbb{Z}^2$ with a portion of its connectedness graph shown in the following figure:

Theorem 1. The Khalimsky plane is homeomorphic to the quotient topological space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the coordinates.

Let $v$ be the Alexandroff $T_4$-topology on $\mathbb{Z}^2$ with a portion of its connectedness graph shown in the following figure:

Theorem 2. The Marcus-Wyse plane is homeomorphic to the quotient topological space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the coordinates with respect to the diagonal axes.

Theorem 3. $(\mathbb{Z}^2, v)$ is homeomorphic to the quotient topological space of $(\mathbb{Z}^2, w)$ given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the coordinates.
By a closure space we understand a set endowed with a Čech closure operator \([1]\), i.e., a closure operator fulfilling all the Kuratowski closure axioms with a possible exception of idempotency. Let \(u\) be the Alexandroff \(T_0\)-closure operator on \(\mathbb{Z}^2\) with a portion of its connectedness graph shown in the following figure where the points that are neither closed nor open are boxed:

\[ \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} \]

**Theorem 4.** \((\mathbb{Z}^2, u)\) is homeomorphic to the quotient closure space of \((\mathbb{Z}^2, w)\) given by the decomposition indicated by the dashed lines in the following figure. Such a homeomorphism is obtained by assigning to every class of the decomposition its center point expressed in the coordinates with respect to the diagonal axes.

3. Digital Jordan curve theorems

The following result is proved in \([7]\):

**Theorem 5.** Every cycle in the graph (with the vertex set \(\mathbb{Z}^2\)) a portion of which is shown in the following figure is a Jordan curve in \((\mathbb{Z}^2, w)\):

\[ \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} \]

Theorems 1-5 enable us to identify Jordan curves among the simple closed curves in the Khalimsky and Marcus-Wyse planes, in \((\mathbb{Z}^2, v)\) and in \((\mathbb{Z}^2, u)\). For example, in addition to the digital Jordan curve theorems proved in \([7]\), we get

a) Let \(D\) be a simple closed curve in \((\mathbb{Z}^2, v)\) with the property that, for each point \((x, y) \in D\), there exists \(k \in \mathbb{Z}\) such that \(y = x + 4k + 2\) or \(y = 4k + 2 - x\). Then \(D\) is a Jordan curve in \((\mathbb{Z}^2, v)\).

b) Every simple closed curve in \((\mathbb{Z}^2, u)\) that is a cycle in the graph a portion of which is shown in the following figure is a Jordan curve in \((\mathbb{Z}^2, u)\):

\[ \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array} \]

References


