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Topological monsters in $\mathbb{Z}^3$: A non-exhaustive bestiary

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1. Introduction

Simple points in $\mathbb{Z}^n$, and especially in $\mathbb{Z}^3$ [1], are the basis of several topology-preserving transformation methods proposed for image analysis (segmentation, skeletonisation, ...). Most of these methods rely on the assumption that the - iterative or parallel - removal of simple points from a discrete object X necessarily leads to a *globally minimal* topologically equivalent sub-object of X (*i.e.* a subset $Y \subseteq X$ which is topologically equivalent to X and which does not strictly include another set $Z \subset Y$ topologically equivalent to X). This is however false in $\mathbb{Z}^3$, and more generally in $\mathbb{Z}^n$ ($n \geq 3$). We illustrate this fact by presenting some *topological monsters*, *i.e.* some objects of $\mathbb{Z}^3$ only composed of non-simple points, but which could however be reduced without altering their topology.

2. Some topological monsters

2.1 Classical monsters...

The most famous example of topological monster is the Bing's house with two rooms [2] which corresponds, in $\mathbb{Z}^3$, to a class of simply connected objects which can not be reduced to singletons by iterative removal of simple points, since they do not contain such points. Bing's houses might be considered as artificial constructions, which are - generally - unlikely to appear in real images, because of their complex shape (quite symmetric, self-bent configuration), and their size (the smallest one contains more than 130 voxels). However, there exists a large (actually infinite) class of objects, some of them being much less complex than Bing's houses, presenting similar properties. It has to be noticed that these objects do not present any specific conditions on their numbers of connected components, holes and cavities. Some of them are described hereafter.

2.2 ...and less classical ones

In this subsection, four topological monsters are described. All of them are considered in a 26-adjacency framework (although there also exist topological monsters in 6-adjacency).

They have been chosen for their small size, their distinct topological properties, and their - relatively - uncomplex shape, in order to convince the reader that topological monsters are not necessarily improbable and/or easy-to-detect objects, and that they can frequently appear in applications involving real images. These four objects, denoted X_i ($i = 1$ to 4), are illustrated in 3D view in Figure 1, and in 2D view in Figure 2. Their main properties are listed in Table 1.

As stated previously, all objects X_1 to X_4 can not be reduced by removal of 26-simple points, since they do not contain any such points.

The first object (X_1) is composed of 32 voxels, which are all non-simple, as $\forall x \in X_1, |C_6^x[N_{18}^*(x) \cap \overline{X_1}]| > 1$ (see [1] for notations and 26-simple point characterisation).

The second object (X_2) is composed of 33 voxels. 32 voxels $x \in X_2$ are such that $|C_6^x[N_{18}^*(x) \cap \overline{X_2}]| > 1$, while the last one (and also its two neighbours) verifies $|C_{26}^x[N_{26}^*(x) \cap X_2]| > 1$. All of them are then non-simple. Note that X_2 is nearly identical to X_1: there is only one supplemental voxel creating one 6-hole/26-cycle. This illustrates the fact that a topological monster can present any topology.

The third (resp. fourth) object (X_3) (resp. X_4) has not been obtained from a simply connected topological monster. 10 (resp. 2) voxels $x \in X_3$ (resp. X_4) verify $|C_6^x[N_{18}^*(x) \cap \overline{X_3}$ (resp. $\overline{X_4}$) $]| > 1$, while the other 8 (resp. 14) ones verify $|C_{26}^x[N_{26}^*(x) \cap X_3$ (resp. X_4) $]| > 1$.

3. Discussion

All objects presented in the last subsection are included in a 6 voxel-width cubical bounding box, and are composed of relatively few voxels. Moreover, their shape is not quite complex. Consequently, the probability of appearance of such objects (or similar ones) during a simple point removal process is not negligible (this assertion is strengthened by the fact that X_1 has been obtained by removing simple points from a 5^3 cube in a random order).

From an experimental point of view, the existence of topological monsters implies the potential inability of methods based on simple point removal to com-

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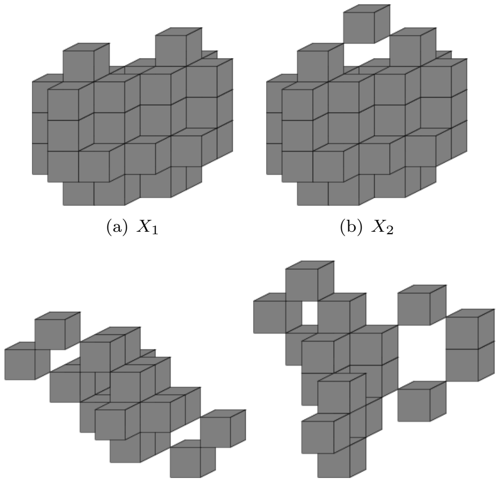
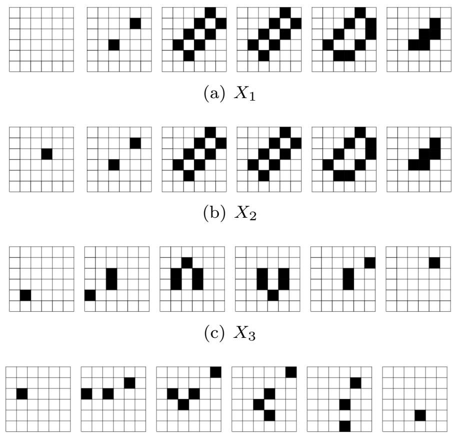
pute globally minimal skeletons, and the appearance of undesired structures in skeletonisation or, more generally, reduction procedures based on the same strategy. These structures could be assimilated to "topological artifacts", by similarity to the terminology of signal-based image processing methods.

It has been observed that the appearance of such structures is favoured by the use of monotonic (*i.e.* reduction or growth) procedures. Indeed, non-monotonic procedures can break artifacts, since they can remove/add voxels which have been previously added/removed, thus generating new simple points where there no longer existed any. The appearance of topological artifacts is also favoured by the use of sophisticated heuristics for choosing the voxels to remove/add. As an example, methods removing voxels by successive layers from the object border (as distance map-based skeletonisation) are unlikely to generate such artifacts, by opposition to methods removing voxels according to intensity in grey-level images [3], and leading to sometimes complex shapes.



(a) X_1 (b) X_2 (c) X_3 (d) X_4

Figure 1. 3D view of four topological monsters X_i ($i = 1$ to 4).



(a) X_1

(b) X_2

(c) X_3

(d) X_4

Figure 2. 2D view of X_i ($i = 1$ to 4).

Table 1. Main properties of X_i ($i = 1$ to 4). $|X_i|$: size (number of voxels) of X_i; $|K(X_i)|$: size of the smallest subset(s) topologically equivalent to X_i; $b_0(X_i)$ (resp. $b_1(X_i)$, resp. $b_2(X_i)$): number of (26-)connected components (resp. (6-)holes, resp. (6-)cavities) of X_i.

X_i	$\|X_i\|$	$\|K(X_i)\|$	$b_0(X_i)$	$b_1(X_i)$	$b_2(X_i)$
X_1	32	1	1	0	0
X_2	33	5	1	1	0
X_3	18	9	1	2	0
X_4	16	14	1	3	0

4. Conclusion

In order to dispose of topological artifacts, especially in monotonic procedures, a solution could consist in generalising the notion of simple points to larger sets, leading to a notion of *simple sets*. The smallest non-trivial and non-unitary ones are composed of two voxels, both being non-simple. It has to be noticed that all topological monsters presented in this paper include at least one such simple set, the removal of which enables to further remove simple points until obtaining a globally minimal topologically equivalent sub-object. First results about the characterisation of such simple sets will be presented soon [4].

References

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