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Mathematical morphology for two valued gray-scale images with undefined information

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1. Introduction

In this work we propose an algebraic model for two valued (l and L) gray-scale images with undefined information (uncertainty) for some pixels. The approach here provides the basis for an extension to multi-scale images, where an uncertainty will be coded by an interval $[l_1, l_2]$. For example, if at some coordinate x there is an uncertainty if the pixel value is 50 or 51, then it will be coded as the mapping $x \mapsto [50, 51]$. The coding of such uncertainty by just considering a new gray-scale λ to represent the uncertainty, does not solve the problem, since λ would be used to codify different situations; for example, $x_1 \mapsto [50, 51]$ and $x_2 \mapsto [75, 76]$ would be coded as $x_1 \mapsto \lambda$ and $x_2 \mapsto \lambda$, leading to a loss of information. Such ideas belong to the notion of interval image and can be seen at [5, 6].

This interval approach, restricted to images with gray-scales in $\{l, L\}$, leads to pixels values in $\{[l, l], [l, L], [L, L]\}$; in other words interval $[l, L]$ is the uncertainty between precise values $[l, l]$ and $[L, L]$.

The type of images we consider in this work is of the form $f : E \to \{[l, l], [l, L], [L, L]\}$, which can be seen as ternary images. We could treat the set of pixel values $\{[l, l], [l, L], [L, L]\}$ as three gray-scales as in standard gray-scale morphology, but this would imply at least a change of spaces (between intervals and gray-scales). Our purpose is then to define the basic operators such as dilations and erosions directly for the interval setting.

These images constitute a pseudo-Boolean algebra (see [7]), whose presence of $[l, L]$ will have a strong influence upon the negation of images as well as upon the morphological operators. The reader should not confuse the value of undefined information as a third intensity value, instead it should be interpreted as a numerical value that represents no information about the pixel intensity.

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2. Algebraic structure of images with undefined information

Let $\Omega = \{[l, l], [l, L], [L, L]\}$, with $[l, l] \leq [l, L] \leq [L, L]$, where the operations $\vee$ and $\wedge$ are defined according to Table 1. It is not difficult to show that

Table 1. Binary operations $\vee$ and $\wedge$ on Ω.

$\vee$	$[L, L]$	$[l, l]$	$[l, L]$
$[L, L]$	$[L, L]$	$[L, L]$	$[L, L]$
$[l, l]$	$[L, L]$	$[l, l]$	$[l, L]$
$[l, L]$	$[L, L]$	$[l, L]$	$[l, L]$

$\wedge$	$[L, L]$	$[l, l]$	$[l, L]$
$[L, L]$	$[L, L]$	$[l, l]$	$[l, L]$
$[l, l]$	$[l, l]$	$[l, l]$	$[l, l]$
$[l, L]$	$[l, L]$	$[l, l]$	$[l, L]$

the structure $\langle \Omega, \vee, \wedge, [L, L], [l, l] \rangle$ is a distributive complete lattice with relative pseudo-complement. Table 2 shows the operations of negation, pseudo-complement ($\wedge$-complement) and $\vee$-complement on Ω. With the introduction of $[l, L]$ we observe that Ω

Table 2. Negation, $\wedge$-complement, $\vee$-complement on Ω.

$\sim$	$[l, l]$	$[l, L]$	$[L, L]$
	$[L, L]$	$[l, L]$	$[l, l]$

$\neg$	$[l, l]$	$[l, L]$	$[L, L]$
	$[L, L]$	$[l, l]$	$[l, l]$

$-$	$[l, l]$	$[l, L]$	$[L, L]$
	$[L, L]$	$[L, L]$	$[l, l]$

is not a Boolean algebra, but a pseudo-Boolean algebra. An operation that changes exact values, that is, changes the elements $[l, l]$ or $[L, L]$ to their dual values will be called **reversion**. We observe that the algebraic structure on Ω provides three reversion operations (see Table 2). Thus, $\langle \Omega^E, \bigvee, \bigwedge, [l, l], [L, L] \rangle$ is

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a complete pseudo-Boolean algebra, with three possible reversion operations: $\sim f$, $\neg f$, and $\overline{f}$.

3. Morphology for undefined images

An undefined image is a function of the form $f : E \to \Omega$, where E is a finite set of coordinates and Ω is the previous lattice. The set of ternary images is denoted by Ω^E. The theory of mathematical morphology is based on lattice theory [1, 4] and aims the topological transformations of images. In this section we will define the elementary morphological operators on ternary images. To simplify notation, we use $\mathcal{T}$ instead of Ω^E.

To define the morphological operators we need to introduce the concepts of translation and Minkowski operations. Notice that our structuring elements are also ternary images. Given two ternary images f_A and f_B, let $B = \{u \in E : f_B(u) \neq [l, l]\}$. The **translation of f_A by a not null coordinate of** f_B, $u \in B$, is the function $f_{A+u} : E \to \Omega$, where:

$$f_{A+u}(x) = f_A(x - u)\,.$$

The Minkowski addition and subtraction are defined respectively by

$$f_A \oplus f_B = \bigvee_{u \in B} f_{A+u}, \qquad (1)$$

$$f_A \ominus f_B = \bigwedge_{u \in B} f_{A-u}\,. \qquad (2)$$

We know that an operator is a **dilation**, if for all family of images $\{f_i\} \subseteq \mathcal{T}$, $\psi(\sup\{f_i\}) = \sup \psi(\{f_i\})$, and it is an **erosion**, if $\psi(\inf\{f_i\}) = \inf \psi(\{f_i\})$.

Definition 1. The **dilation** and the **erosion** of f_A with respect to B, denoted respectively by $\delta_B(f_A)$ and $\varepsilon_B(f_A)$, are defined respectively by

$$\delta_B(f_A)(x) = \bigvee_{u \in B} f_A(x - u)$$

$$\varepsilon_B(f_A)(x) = \bigwedge_{u \in B} f_A(x + u)\,.$$

In other words, $\delta_B(f_A)(x) = (f_A \oplus f_B)(x)$ and $\varepsilon_B(f_A)(x) = (f_A \ominus f_B)(x)$, where the ternary image f_B is a structuring element.

Proposition 1. *The functions $\delta_B, \varepsilon_B : \mathcal{T} \to \mathcal{T}$ are a dilation and an erosion, respectively.*

Proof. $\bigvee \delta_B(\{f_i\}_{i\in I})(x) = \bigvee \delta_B(\{f_A : f_A \in \{f_i\}_{i\in I}\})(x) = \bigvee\{\delta_B(f_A) : f_A \in \{f_i\}_{i\in I}\}(x) = \bigvee\{\delta_B(f_A)(x) : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_{A+u} : u \in B\}(x) : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_{A+u}(x) : u \in B\} : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_A(x-u) : u \in B\} : f_A \in \{f_i\}_{i\in I}\} = \bigvee\{\bigvee\{f_A(x-u) : f_A \in \{f_i\}_{i\in I}\} : u \in B\}$ (by associativity and commutativity of "$\bigvee$"). Moreover, making $h_A = \bigvee\{f_i\}_{i\in I}$, $\delta_B(h_A)(x) = \bigvee_{u\in B}\{h_{A+u}\}(x) = \bigvee\{h_{A+u} : u \in B\}(x) = \bigvee\{h_{A+u}(x) : u \in B\} = \bigvee\{h_A(x-u) : u \in B\} = \bigvee\{\bigvee\{f_i\}_{i\in I}(x-u) : u \in B\} = \bigvee\{\bigvee\{f_A : f_A \in \{f_i\}_{i\in I}\}(x-u) : u \in B\} = \bigvee\{\bigvee\{f_A(x-u) : f_A \in \{f_i\}_{i\in I}\} : u \in B\}$.

Therefore, for all x, $\bigvee \delta_B(\{f_i\}_{i\in I})(x) = \delta_B(\bigvee\{f_i\}_{i\in I})(x)$; i.e., $\bigvee \delta_B(\{f_i\}_{i\in I}) = \delta_B(\bigvee\{f_i\}_{i\in I})$.

By duality we prove that ε_B is an erosion. □

In the same way we combine basic Boolean functions to define complex combinational circuits, we can also combine the previous operators to build more complex operators to deal with ternary images. For instance, two important morphological operators are **openings** and **closings**, commonly used for *noise filtering*. They can be defined as compositions of erosions and dilations, with one of the reversion operations. Other Boolean operations like XOR, NAND and NOR can also be defined on ternary images, also with one of the reversion operations. Thus, although being essentially equivalent to gray-scale morphology from an algebraic point of view, the formulation presented here allows the modeling of uncertainty of pixel values. Moreover, considering different reversion operations, different results can be obtained.

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