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Centres of maximal balls extracted from a fuzzy distance transform

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1. Introduction

During the last 10 years or so, there has been a large movement within the field of computerised image analysis to avoid crisp segmentation into interesting and non-interesting regions and instead using a fuzzy approach, where points are assigned membership values related to the degree of belongingness to the structure of interests. The segmented fuzzy object is used in the subsequent analysis. By this, the analysis is made less dependent on small changes in the border, which may be imposed by a crisp segmentation, as well as more scale invariant. See, e.g., [7]. The idea of partial means to a set was introduced in [8] and, in an image analysis context in [4].

We present an extension of the concept of centres of maximal balls (CMBs) for binary images, [1], to a fuzzy framework. CMBs are in the binary case extracted from the distance transform (DT) of the object by simple value comparison. In the fuzzy case, we consider the grey-weighted distance in [3], for which the mean of the grey-levels of two neighbouring points is multiplied by the spatial distance between them. This has been put into a theoretical framework and denoted the fuzzy distance transform (FDT) in [5]. By extracting centres of maximal balls from a FDT, we obtain a more stable result with respect to size and fuzziness. This is of interest, e.g. for fuzzy distance based skeletonization.

2. Notions

Let X be the reference set, then a *fuzzy subset* $\mathcal{A}$ of X is defined as a set of ordered pairs $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \mid x \in X\}$, where $\mu_{\mathcal{A}} : X \to [0,1]$ is the *membership function* of $\mathcal{A}$ in X. A 2D *fuzzy digital object* $\mathcal{O}$ is a fuzzy subset defined on $\mathbb{Z}^2$, i.e., $\mathcal{O} = \left\{(p, \mu_{\mathcal{O}}(p)) \mid p \in \mathbb{Z}^2\right\}$, where $\mu_{\mathcal{O}} : \mathbb{Z}^2 \to [0,1]$. A pixel p belongs to the support of $\mathcal{O}$ if $\mu_{\mathcal{O}}(p) > 0$.

The fuzzy distance between two points p and q in a fuzzy object $\mathcal{O}$ is defined as being the length of the shortest path between p and q. In [3, 5], the fuzzy distance $d_{\mathcal{O}}^{\langle 1,\sqrt{2}\rangle} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ between p and q in $\mathcal{O}$ is set to

$$d_{\mathcal{O}}^{\langle 1,\sqrt{2}\rangle} = \min_{\langle p=p_1,\ldots,p_m=q\rangle} \sum_{i=1}^{m-1} \tfrac{1}{2}\left(\mu_{\mathcal{O}}(p_i) + \mu_{\mathcal{O}}(p_{i+1})\right) \cdot w(p_{i+1} - p_i),$$

where $w(p_{i+1} - p_i)$ is the spatial Euclidean distance between p_i and p_{i+1}, i.e., 1 if p_i and p_{i+1} are edge neighbours and $\sqrt{2}$ if they are vertex neighbours.

A fuzzy distance more stable under rotation can be obtained by using other weights, e.g. the $\langle 3,4\rangle$ distance, where 3 is the weight to an edge neighbour and 4 the weight to a vertex neighbour [2]. The fuzzy $\langle 3,4\rangle$ distance, $d_{\mathcal{O}}^{\langle 3,4\rangle}$, is the one that will be used here. In a *fuzzy distance transform* (FDT), a point $p \in \mathcal{O}$ is assigned to its fuzzy $\langle 3,4\rangle$ distance from the complement of the support of $\mathcal{O}$, $D^{\mathrm{F}}(\mathcal{O}, \langle 3,4\rangle, p)$. We use D^{F} to denote the value of a point in a FDT and $d_{\mathcal{O}}$ for the distance function used to calculate a FDT.

3. Centres of maximal balls

Given a (crisp) object $\mathcal{O} \in \mathbb{Z}^n$, a distance function $d : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$, and the corresponding DT computed on $\mathcal{O}$, the distance value of a point $\mathbf{c} \in \mathcal{O}$ in the DT can be interpreted as the radius r of a ball $\mathcal{B}(d, \mathbf{c}, r) = \{\mathbf{x} \in \mathbb{Z}^n \mid d(\mathbf{c}, \mathbf{x}) \leq r\}$ such that $\mathcal{B} \subseteq \mathcal{O}$ and $\mathcal{B}(d, \mathbf{c}, r+\epsilon) \not\subseteq \mathcal{O}$, $\epsilon > 0$. Let $\mathrm{DT}\mathcal{B} = \{\mathcal{B}_i(d, \mathbf{c_i}, r_i) \mid i = 1, \ldots m\}$, where $\mathbf{c}_i$ are all the points in $\mathcal{O}$, r_i are the respective values in the DT, and m is the number of points in $\mathcal{O}$. Hence, $\mathcal{O} = \bigcup_{i=1}^{m} \mathcal{B}_i$, where $\mathcal{B}_i \in \mathrm{DT}\mathcal{B}$.

$\mathcal{B}^{\mathrm{M}}$ is denoted a *maximal ball* and $\mathbf{c}^{\mathrm{M}}$ a *centre of a maximal ball*, if for all $\mathcal{B}_i \in \mathrm{DT}\mathcal{B}$, $\mathcal{B}_i \not\supset \mathcal{B}^{\mathrm{M}}$, $i = 1, \ldots m$. Thus, $\bigcup_{i=1}^{k} \mathcal{B}_i^{\mathrm{M}}$ is equal to $\mathcal{O}$. This means that $\mathcal{O}$ can be represented by its set of centres of maximal balls, denoted $\mathrm{CMB}(\mathcal{O}, D)$, as $\mathcal{O}$ can be recovered from $\mathrm{CMB}(\mathcal{O}, D)$.

$\mathrm{CMB}(\mathcal{O}, D)$ can be identified in one scan over the DT by value comparison [1]. This is due to the fact that $\mathrm{CMB}(\mathcal{O}, D)$ consists of the points in the DT which do not propagate distance information to neighbouring points, i.e., $\mathrm{CMB}(\mathcal{O}, D)$ consists of "local maxima". Considering $\mathcal{O} \in \mathbb{Z}^2$ and the $\langle 3,4\rangle$ distance, a pixel p belongs to $\mathrm{CMB}(\mathcal{O}, \langle 3,4\rangle)$ if, for all element n_i, $i = 1, \ldots, 8$ in the neighbourhood, given in local coordinates around p, with their respective weights $w(n_i) \in \{3,4\}$, if

$$D(\mathcal{O}, \langle 3,4\rangle, p+n_i) < D(\mathcal{O}, \langle 3,4\rangle, p) + w(n_i),$$

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where $D(\mathcal{O}, \langle 3,4\rangle, p)$ is the distance value of pixel p found in the $\langle 3,4\rangle$ DT of $\mathcal{O}$. $\text{CMB}\,(\mathcal{O}, D)$ is a compact representation of $\mathcal{O}$. It is also often used as non removable points in skeletonization to guarantee full reversibility of the object as $\mathcal{O}$ can be recovered from $\text{CMB}\,(\mathcal{O}, D)$ by means of the reverse DT.

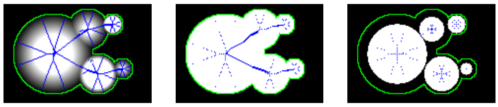
We here extend the concept of centres of maximal balls (the crisp case) to a fuzzy framework by introducing the centres of maximal balls extracted from a FDT, denoted $\text{CMFB}\left(\mathcal{O}, D^{\text{F}}\right)$, where $d_{\mathcal{O}} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R}$ is the underlying fuzzy distance function and D^{F} are the values in the corresponding FDT. Analogous to the crisp case, we define $\text{CMFB}\left(\mathcal{O}, D^{\text{F}}\right)$ to be the points which do not propagate distance information to neighbouring points while calculating the FDT. $\text{CMFB}\left(\mathcal{O}, D^{\text{F}}\right)$ can be detected in one scan over the FDT of $\mathcal{O}$ by value comparison, taking into account also the membership values in $\mathcal{O}$. Considering $\mathcal{O} \in \mathbb{Z}^2$ and the fuzzy $\langle 3,4\rangle$ distance, a pixel p belongs to $\text{CMFB}\,(\mathcal{O}, \langle 3,4\rangle)$ if, for all element n_i, $i = 1, \ldots, 8$ in the neighbourhood, with their respective weights $w(n_i) \in \{3, 4\}$,

$$\begin{aligned} D^{\text{F}}(\mathcal{O}, \langle 3,4\rangle, p + n_i) < D^{\text{F}}(\mathcal{O}, \langle 3,4\rangle, p) + \\ + \tfrac{1}{2}\left(\mu_{\mathcal{O}}(p) + \mu_{\mathcal{O}}(p + n_i)\right) \cdot w(n_i), \end{aligned}$$

where $D^{\text{F}}(\mathcal{O}, \langle 3,4\rangle, p)$ is the distance value at p in the $\langle 3,4\rangle$ FDT. We remark that from $\text{CMFB}\,(\mathcal{O}, \langle 3,4\rangle)$, the support of $\mathcal{O}$ can be recovered by means of the reverse fuzzy distance transform [6].

To show the advantages of a fuzzy approach compared to a crisp one, we use the example in Figure 1. The fuzzy object $\mathcal{O}_1$ (Figure 1(a)) is composed of balls of different size and levels of fuzziness. Its $\text{CMFB}\,(\mathcal{O}_1, \langle 3,4\rangle)$ is shown in blue. Two crisp objects, $\mathcal{O}_2$ and $\mathcal{O}_3$, created by simple thresholding of $\mathcal{O}_1$, are shown in Figures 1(b) and 1(c), respectively, with their $\text{CMB}\,(\mathcal{O}_2, \langle 3,4\rangle)$ and $\text{CMB}\,(\mathcal{O}_3, \langle 3,4\rangle)$. The support of $\mathcal{O}_1$ is outlined in green for comparison. As can be seen, $\text{CMFB}\,(\mathcal{O}_1, \langle 3,4\rangle)$ exhibits similar configuration for all the circular subparts, independently of size and level of fuzziness, whereas the crisp representations $\text{CMB}\,(\mathcal{O}_2, \langle 3,4\rangle)$ and $\text{CMB}\,(\mathcal{O}_3, \langle 3,4\rangle)$ are rather unstable representations, varying with the shape of the crisp object and its parts. This indicates that a more stable result, with respect to both size and fuzziness, can be achieved using a fuzzy approach. In Figure 2, two fuzzy objects, $\mathcal{O}_4$ and $\mathcal{O}_5$, are shown with their $\text{CMFB}\,(\mathcal{O}_i, \langle 3,4\rangle)$ in blue and the support of $\mathcal{O}_i$ outlined in green. $\mathcal{O}_4$ and $\mathcal{O}_5$ illustrate the fact that by the described fuzzy approach, the internal grey-level structure is enhanced, something which cannot be achieved by using a crisp approach.

The above result is applicable to FDTs based on other fuzzy distance functions. Future work will be to use the centres of maximal balls extracted from a FDT in fuzzy distance based skeletonization.



(a) (b) (c)

Figure 1. (a) A fuzzy object $\mathcal{O}_1$ with $\text{CMFB}\,(\mathcal{O}_1, \langle 3,4\rangle)$ shown in blue. (b) A crisp object $\mathcal{O}_2$ with $\text{CMB}\,(\mathcal{O}_2, \langle 3,4\rangle)$. (c) A crisp object $\mathcal{O}_3$ with $\text{CMB}\,(\mathcal{O}_3, \langle 3,4\rangle)$. The support of $\mathcal{O}_1$ outlined in green.

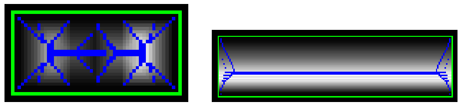


Figure 2. Fuzzy objects $\mathcal{O}_i$ with $\text{CMFB}\,(\mathcal{O}_i, \langle 3,4\rangle)$ in blue and the support of $\mathcal{O}_i$ outlined in green ($i = 4, 5$).

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