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Mathematical morphology on bipolar fuzzy sets

Isabelle Bloch

Ecole Nationale Supérieure des Télécommunications (GET - Télécom Paris), CNRS UMR 5141 LTCI, Paris, France - `Isabelle.Bloch@enst.fr`

1. Introduction

In many domains, it is important to be able to deal with bipolar information [5]. Positive information represents what is granted to be possible (for instance because it has already been observed or experienced), while negative information represents what is impossible (or forbidden, or surely false). This domain has recently motivated work in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been proposed [5]. When dealing with spatial information, in image processing or for spatial reasoning applications, this bipolarity also occurs. For instance, when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indetermination. This corresponds to the idea that the union of positive and negative information does not cover all the space.

To our knowledge, bipolarity has not been much exploited in the spatial domain. The above considerations are the motivation for the present work, which aims at filling this gap by proposing formal models to manage spatial bipolar information. Additionally, imprecision has to be included, since it is an important feature of spatial information, related either to the objects themselves or to the spatial relations between them. More specifically, we consider bipolar fuzzy sets, and propose definitions of mathematical morphology operators (dilation and erosion) on these representations. To our knowledge, this is a new contribution.

2. Lattice structure and algebraic bipolar fuzzy dilations and erosions

Let $\mathcal{S}$ be the underlying space (the spatial domain for spatial information processing). A bipolar fuzzy set on $\mathcal{S}$ is defined by a pair of functions (μ, ν) such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. Let us consider the set of pairs of numbers (a, b) in $[0, 1]$ such that $a + b \leq 1$. This set is a complete lattice, for the partial order defined as [3]: $(a_1, b_1) \preceq (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \geq b_2$. The greatest element is $(1, 0)$ and the smallest element is $(0, 1)$. The supremum and infimum are respectively defined as: $(a_1, b_1) \vee (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2))$, and $(a_1, b_1) \wedge (a_2, b_2) = (\min(a_1, a_2), \max(b_1, b_2))$. The partial order $\preceq$ induces a partial order on the set of bipolar fuzzy sets: $(\mu_1, \nu_1) \preceq (\mu_2, \nu_2)$ iff $\forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x)$ and $\nu_1(x) \geq \nu_2(x)$. If $\mathcal{B}$ denotes the set of bipolar fuzzy sets on $\mathcal{S}$, $(\mathcal{B}, \preceq)$ is a complete lattice.

Once we have a complete lattice, it is easy to define algebraic dilations and erosions on this lattice, as operations that commute with the supremum and the infimum, respectively, as in classical mathematical morphology [6]. We can then derive the usual properties based on the algebraic framework of complete lattices and of adjunctions.

3. Morphological bipolar fuzzy dilations and erosions

Next, if $\mathcal{S}$ is an affine space (or at least a space on which translations can be defined), the general principle underlying morphological erosions is to translate the structuring element at every position in space and check if this translated structuring element is included in the original set [8]. This principle has also been used in the main extensions of mathematical morphology to fuzzy sets (see e.g. [2, 7]). Similarly, defining morphological erosions of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion between bipolar fuzzy sets. Such inclusion degrees have been proposed in the context of intuitionistic fuzzy sets in [4]. With our notations, a degree of inclusion of a bipolar fuzzy set (μ', ν') in another bipolar fuzzy set (μ, ν) is defined as: $\inf_{x \in \mathcal{S}} I((\mu'(x), \nu'(x)), (\mu(x), \nu(x)))$ where I is an implication operator. Two types of implication are used in [4], one derived from an intuitionistic (or bipolar) t-conorm $\perp$, and one derived

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from a residuation principle from an intuitionistic t-norm $\top$. We developed morphological dilations and erosions for bipolar fuzzy sets based on these two types of implication and on properties of duality and/or adjunction.

The erosion of any (μ,ν) in $\mathcal{B}$ by a bipolar fuzzy structuring element (μ_B,ν_B) (in $\mathcal{B}$) is defined from an implication I as: $\forall x \in \mathcal{S}, \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \inf_{y\in\mathcal{S}} I((\mu_B(y-x),\nu_B(y-x)),(\mu(y),\nu(y)))$.

The dilation of any (μ,ν) in $\mathcal{B}$ by (μ_B,ν_B) is defined from erosion by duality as: $\delta_{(\mu_B,\nu_B)}((\mu,\nu)) = c[\varepsilon_{(\mu_B,\nu_B)}(c((\mu,\nu)))]$, where c is a bipolar complementation (e.g. the standard negation defined as $c(a,b)=(b,a)$).

Using a residual implication for the erosion for a bipolar t-norm $\top$, the bipolar fuzzy dilation, adjoint of the erosion, is defined as:

$$\begin{aligned}\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \\ &= \inf\{(\mu',\nu')(x), (\mu,\nu)(x) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu',\nu'))(x)\} \\ &= \sup_{y\in\mathcal{S}} \top((\mu_B(x-y),\nu_B(x-y)),(\mu(y),\nu(y))).\end{aligned}$$

It has been shown in [4] that adjoint bipolar t-norms and t-conorms are all derived from the Lukasiewicz operators, using a continuous bijective permutation on $[0,1]$. Hence equivalence between both approaches can be achieved only for this class of operators. The bipolar Lukasiewicz t-norm is defined as $\top_W((a_1,b_1),(a_2,b_2)) = (\max(0,a_1+a_2-1),\min(1,b_1+1-a_2,b_2+1-a_1))$, while the bipolar Lukasiewicz t-conorm is defined as $\perp_W((a_1,b_1),(a_2,b_2)) = (\min(1,a_1+1-b_2,a_2+1-b_1),\max(0,b_1+b_2-1))$.

4. Properties and interpretations

These definitions of morphological bipolar fuzzy dilations and erosions are shown to have nice properties, with respect to the usual properties of mathematical morphology, as detailed in [1]:

- they actually provide bipolar fuzzy sets;
- in case the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1-\mu$ and $\nu_B = 1-\mu_B$), the definitions lead to the usual definitions of fuzzy dilations and erosions (hence they are also compatible with classical morphology in case μ and μ_B are crisp);
- they commute respectively with the supremum and the infinum of the lattice $(\mathcal{B},\preceq)$;
- the bipolar fuzzy dilation is extensive (i.e. $(\mu,\nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu))$) and the bipolar fuzzy erosion is anti-extensive (i.e. $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \preceq (\mu,\nu)$) if and only if $(\mu_B,\nu_B)(0) = (1,0)$, where 0 is the origin of the space $\mathcal{S}$ (i.e. the origin completely belongs to the structuring element, without any indetermination);
- if the dilation is defined from a t-representable t-norm, the following iterativity property holds: $\delta_{(\mu_B,\nu_B)}(\delta_{(\mu'_B,\nu'_B)}((\mu,\nu))) = \delta_{(\delta_{\mu_B}(\mu'_B),1-\delta_{(1-\nu_B)}(1-\nu'_B))}((\mu,\nu))$.

These definitions have also interesting interpretations. For instance, the dilation of (μ,ν) by a bipolar structuring element (μ_B,ν_B), based on t-representable bipolar t-norms (i.e. composed of usual t-norms t and t-conorms T: $\top((a_1,b_1),(a_2,b_2)) = (t(a_1,a_2),T(b_1,b_2))$), is expressed as: $\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = (\sup_{y\in\mathcal{S}} t(\mu_B(x-y),\mu(y)),\inf_{y\in\mathcal{S}} T((\nu_B(x-y),\nu(y)))$. The first term (membership function) is exactly the fuzzy dilation of μ by μ_B, while the second one (non-membership function) is the fuzzy erosion of ν by $1-\nu_B$, according to the definitions of [2].

Examples will illustrate the effect of these operations on spatial bipolar fuzzy sets representing positive and negative information on the position of some objects.

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