Determination of the Minimum Distance Between Symbols of the Two Non-Orthogonal M-QAM Carriers

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Abstract— **Multicarrier communication systems have become ubiquitous, mainly due to the popularization of orthogonal frequency division multiplexing (OFDM), in which carriers are separated in frequency by the inverse of the symbol duration. Recently, more spectrally efficient modulations based on non-orthogonal carriers (non-OFDM) have been put forward and shown numerically to have the same performance as OFDM employing up to 40% less bandwidth. This work addresses the problem of analytically deriving the minimum frequency separation which does not affect the minimum distance between multicarrier symbols. In doing so, it shows that the probability of error remains unaffected up to a certain degree of spectral superposition of the carriers, so that the performance of non-OFDM in terms of bit error rate (BER) remains the same as OFDM. Simulations and comparisons to previous numerical results are used to illustrate this conclusion.**

Keywords— **Mutlicarrier communication, OFDM, non-OFDM, non-orthogonal multicarrier communication**

I. Introduction

Orthogonal frequency division multiplexing (OFDM) [1] has become the preferred method for communication over wireless broadband channels. This multicarrier scheme is adopted in a wide range of systems, such as 3GPP Long-Term Evolution (LTE) [2], Worldwide Interoperability for Microwave Access (WiMax) [3], and Digital Video Brodcasting (DVB) [4]. In all these applications, the frequency separation between carriers is the inverse of symbol duration which is the necessary condition for the orthogonality of the carriers [5].

Recently, non-orthogonal multicarrier communication systems (non-OFDM)—i.e., systems in which the carriers separation is smaller than OFDM—have been proposed [6]–[8]. Numerical results have shown that their performance in terms of bit error rate (BER) is the same as OFDM up to a certain degree of spectral superposition, allowing spectral efficiency gains of up to 40% [6], [7]. An analytical determination of the relationship between the degree of spectral overlapping and BER, however, remains an open problem.

This work will study the case of two non-orthogonal carriers and rectangular pulse shape by

- deducing the spectral overlapping bound—valid for any complex constellation—above which the carriers separation does not interfere with the minimum distance between symbols;
- using this bound to determine the conditions under which the BER of non-OFDM is equivalent to that of OFDM;
- supporting these findings by means of numerical simulations and comparisons to previous results in the literature [6], [7].

In the sequel, Section II formulates the problem and introduces the symbol distance measure in non-OFDM systems. Section III then derives the analytical bound for the minimum distance, which is supported by numerical experiments in Section IV. Finally, Section V states the conclusions and perspectives of this work.

II. Problem formulation

An arbitrary constellation $\mathcal{C}$ over the signal space $\mathcal{R}^2$ is a set of symbols $x_m \in \mathbb{C}$, $m = 1, \dots, M$, represented by the band pass signals $s_m(t) = \mathbb{R}\mathrm{e}\{x_m g(t) e^{j2\pi f_0 t}\}$, where f_0 is the carrier frequency, $g(t)$ is a pulse shape function, and $\mathbb{R}\mathrm{e}\{a + jb\} = a$ is the real part of a complex number. For instance, the Quadrature Amplitude Modulation (QAM) with $M = 4$ and rectangular constellation has $\mathcal{C} = \{1 + j, 1 - j, -1 + j, -1 - j\}$.

In multicarrier communication systems, N carriers spectrally separated by Δf are used to transmit elements of $\mathcal{C}$. In this case, a symbol is a vector $\boldsymbol{x}_\ell \in \mathcal{C}^N$, $\ell = 1, \dots, M^N$ that captures the information transmitted over each carrier. The bandpass signal for a sequence of these symbols $\{\boldsymbol{x}(k)\}$, $k \in \mathbb{N}$, is $s(t) = \sum_k \mathbb{R}\mathrm{e}\{\boldsymbol{x}(k)^T \boldsymbol{\psi}(t - kT) g(t - kT)\}$, where $\boldsymbol{\psi}(t) = [\ e^{j2\pi f_0 t} \ \dots \ e^{j2\pi[f_0 + (N-1)\Delta f]t}\ ]^T$ and ^{T} is the transposition operator [5].

Assuming the pulse shape $g(t)$ has length T—$g(t) \neq 0$ only for $t \in [0, T)$—, define $\Delta f T$ as the relative spectral separation between carriers. The smaller the value of $\Delta f T$, the more overlapping there is between subchannels, and the more spectrally efficient the modulation scheme is. As mentioned before, OFDM uses $\Delta f T = 1$, whereas non-OFDM systems use $\Delta f T < 1$ [6]–[8].

A. Probability of error over an AWGN channel

The received signal for an additive white Gaussian noise (AWGN) channel is given by $r(t) = s(t) + v(t)$, where $v(t)$ is a white Gaussian process with zero mean and power

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spectral density $N_0/2$. The maximum likelihood receiver can then be implemented by minimizing the distance

$$\mathcal{D}_k[r(t), s_\ell(t)] = \int_0^T |r(kT+\tau) - s_\ell(\tau)|^2 \, d\tau \tag{1}$$

over all possible $s_\ell(t) = \mathbb{R}\mathrm{e}\{\boldsymbol{x}_\ell(k)^T \boldsymbol{\psi}(t) g(t)\}$ [5].

The performance of this receiver is evaluated by the probability of error $P_i(k)$, defined as the probability that the k-th symbol $\boldsymbol{x}_i$ is received as $\boldsymbol{x}_j$, $\forall\, j \neq i$. Explicitly

$$P_i(k) = P\left[\bigcup_{j\neq i} e_j(k)\right] \leq \sum_{j\neq i} P[e_j(k)], \tag{2}$$

where $e_j(k)$ represents the event of demodulating $r(\tau)$—$\tau \in [kT, (k+1)T]$—as $\boldsymbol{x}_j$, $P[x]$ is the probability of the event x, and the last inequality yields from the union bound [5]. For both single and multicarrier systems,

$$P[e_j(k)] = Q\left(\sqrt{\frac{\mathcal{D}_k[s_i(t), s_j(t)]}{2N_0}}\right), \tag{3}$$

where $Q(x) = 1 - \Phi(x)$ and $\Phi(x)$ is the cumulative distribution of the standard Gaussian random variable [6], [9]. In the single carrier case, $\mathcal{D}_k$ reduces to the Euclidian distance

$$\mathcal{D}_k[s_i(t), s_j(t)] = d_{ij}^2 = |x_j - x_i|^2, \tag{4}$$

with $x_i, x_j \in \mathcal{C}$. Notice that (4) is independent of k.

The multicarrier case, however, is more intricate since the distance (1) will be time variant in the general case. Indeed, for a rectangular pulse shape of width T—i.e., $g(t)$ is unitary for $t = [0, T)$ and null otherwise—and $\boldsymbol{x}_i, \boldsymbol{x}_j \in \mathcal{C}^N$ one has

$$\mathcal{D}_k[s_i(t), s_j(t)] = D_{ij}^2(k) = [\boldsymbol{x}_j - \boldsymbol{x}_i]^* \boldsymbol{H}(k) [\boldsymbol{x}_j - \boldsymbol{x}_i], \tag{5}$$

where

$$\boldsymbol{H}(k) = \begin{bmatrix} 1 & h_1(k) & \cdots & h_{N-1}(k) \\ h_1^*(k) & 1 & \cdots & h_{N-2}(k) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1}^*(k) & h_{N-2}^*(k) & \cdots & 1 \end{bmatrix},$$

with $h_n(k) = \mathrm{sinc}(n\Delta f T) e^{-jn\phi(k)}$, $\phi(k) = \pi \Delta f T(2k+1)$, and * denoting the conjugate transpose operation [7].

Notice from (5) that, even though $\boldsymbol{H}(k)$ is time variant, $D_{ij}^2(k)$ is a proper norm for $\Delta f T \neq 0$ since $\boldsymbol{H}(k)$ is positive-definite [10]. Also, for $\Delta f T = 1$—the OFDM case—$\boldsymbol{H}(k) = \boldsymbol{I}$, the identity matrix, and (5) reduces to the Euclidian norm. Indeed, it is a well-established result that the probability of error of OFDM is the same as that of a single carrier system [5].

The performance of non-OFDM will be at least as good as OFDM—and therefore single carrier modulation—if

$$\min_{i\neq j} D_{ij}^2(k) \geq \min_{i\neq j} d_{ij}^2 = d_{min}^2, \ \forall\, k, \tag{6}$$

where d_{min} is the minimum distance between symbols in $\mathcal{C}$. The next section determines the minimum value of $\Delta f T$ for which this condition holds.

III. Spectral separation lower bound

Due to space constraints, the following derivations are carried out for $N = 2$ carriers. However, no restriction is imposed on the size—M—or form of $\mathcal{C}$.

Define the difference vector $\boldsymbol{\delta}_{ij} = [\ \delta_{ij,1} \quad \delta_{ij,2}\ ]^T = \boldsymbol{x}_j - \boldsymbol{x}_i$, with $\boldsymbol{x}_i, \boldsymbol{x}_j \in \mathcal{C}^2$. Then, (5) can be written as

$$D_{ij}^2(k) = \|\boldsymbol{\delta}_{ij}\|^2 + 2\,\mathrm{sinc}(\Delta f T)\, \mathbb{R}\mathrm{e}\{\delta_{ij,1}\delta_{ij,2}^* e^{-j\phi(k)}\},$$

which for $\delta_{ij,n} = |\delta_{ij,n}|\, e^{j\theta_{ij,n}}$ yields

$$\begin{aligned} D_{ij}^2(k) = {} & \|\boldsymbol{\delta}\|^2 \\ & + 2\,|\delta_1|\,|\delta_2|\, \mathrm{sinc}(\Delta f T) \cos[\theta_1 - \theta_2 + \phi(k)]. \end{aligned} \tag{7}$$

The symbol vectors indexes i, j were omitted for clarity's sake.

The following theorem determines the minimum relative spectral separation $\Delta f T$ for which condition (6) holds. As shown in Section II-A, this is equivalent to guaranteeing that the probability of error of non-OFDM is the same as that of OFDM—and single carrier modulation—for a given constellation.

Theorem 1: In a multicarrier system composed of $N = 2$ carriers spectrally separated by Δf transmitting symbols from a constellation $\mathcal{C}$ using rectangular-shaped pulse of length T,

$$\min_{i\neq j} D_{ij}^2(k) = d_{min}^2, \ \forall\, k \Leftrightarrow \mathrm{sinc}(\Delta f T) \leq 0.5, \tag{8}$$

or using a Taylor series approximation, $\Delta f T > 0.6033$.

Due to the different aspects involved in the proof of Theorem 1, it has been divided in three parts. First, Lemma 1 studies a particular case of the problem.

Lemma 1: When the difference vector $\boldsymbol{\delta}$ has a vanishing element,

$$\min_{i\neq j} D_{ij}^2(k) = d_{min}^2, \ \forall\, k.$$

Proof. For $\delta_1 = 0$ or $\delta_2 = 0$, $D_{ij}^2(k) = \|\boldsymbol{\delta}\|^2$, $\forall\, k$. Without loss of generality, assume $\delta_1 = 0$, so that $\min_{i\neq j} D_{ij}^2(k) = \min_{\delta_2 \neq 0} \|\delta_2\|^2 = d_{min}^2$.

■

Now, assuming $\delta_n \neq 0$, $n = 1, 2$, (6) is equivalent to

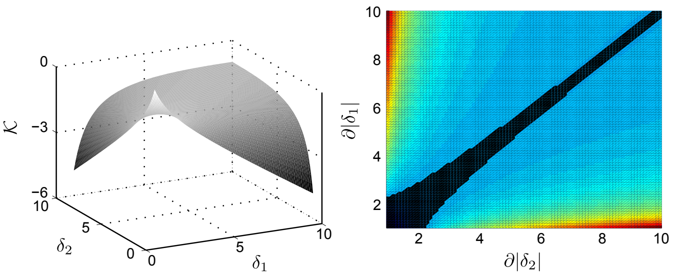
$$\begin{aligned} \cos[\theta_1 - \theta_2 + \phi(k)] &\geq \frac{\mathcal{K}}{\mathrm{sinc}(\Delta f T)} \\ \mathcal{K} &= \frac{d_{min}^2 - \|\boldsymbol{\delta}\|^2}{2\,|\delta_1|\,|\delta_2|}. \end{aligned} \tag{9}$$

It is straightforward to see that $\delta_n \neq 0 \Leftrightarrow \|\boldsymbol{\delta}\|^2 > d_{min}^2 \Leftrightarrow \mathcal{K} < 0$. Lemma 2 and 3 address the limit values of $\mathcal{K}$ and the cosine in (9), which are used to prove the necessary and sufficient condition of Theorem 1.

Lemma 2: For $\delta_n \neq 0$, $n = 1, 2$, $\max_{\boldsymbol{\delta}} \mathcal{K} = -1/2$.

Proof. The partial derivatives of $\mathcal{K}$

$$\begin{aligned} \frac{\partial \mathcal{K}}{\partial\, |\delta_1|} &= \frac{|\delta_2|^2 - d_{min}^2 - |\delta_1|^2}{2\,|\delta_1|^2\,|\delta_2|} \\ \frac{\partial \mathcal{K}}{\partial\, |\delta_2|} &= \frac{|\delta_1|^2 - d_{min}^2 - |\delta_2|^2}{2\,|\delta_1|\,|\delta_2|^2} \end{aligned}$$



(a) $\mathcal{K}$ (b) $\frac{\partial \mathcal{K}}{\partial |\delta_n|}$

Fig. 1. Surface of $\mathcal{K}$ and its partial derivatives

vanish over the hyperbolae

$$\begin{aligned}\frac{\partial \mathcal{K}}{\partial |\delta_1|} &= 0 \Leftrightarrow |\delta_2|^2 - |\delta_1|^2 = d_{min}^2 \\ \frac{\partial \mathcal{K}}{\partial |\delta_2|} &= 0 \Leftrightarrow |\delta_1|^2 - |\delta_2|^2 = d_{min}^2.\end{aligned} \tag{10}$$

The surface of $\mathcal{K}$ and these delimiting hyperbolae are illustrated in Figure 1 for $d_{min}^2 = 4$ (4-QAM with square constellation).

Inside these boundaries (dark region in Fig. 1), both derivatives are negative and $\mathcal{K}$ is a strictly decreasing function. On the other hand, outside these boundaries (light region in Fig. 1) $\mathcal{K}$ will increase (decrease) with relation to $|\delta_1|$ and decrease (increase) with relation $|\delta_2|$.

The derivations are, therefore, split in two cases:

(i) *Outside the hyperbolae*: Without loss of generality, assume $\mathcal{K}$ is a decreasing function of $|\delta_2|$—and hence an increasing function of $|\delta_1|$. Thus, it is maximized over the hyperbolae $|\delta_2|^2 = d_{min}^2 + |\delta_1|^2$ and asymptotically approaches

$$\lim_{|\delta_2|^2 = d_{min}^2 + |\delta_1|^2, |\delta_1| \to \infty} \mathcal{K} = \lim_{|\delta_1| \to \infty} -\frac{|\delta_1|}{\sqrt{|\delta_1|^2 + d_{min}^2}} = -1.$$

The same result is obtained for $|\delta_2| \to \infty$ over the boundary $|\delta_1|^2 = d_{min}^2 + |\delta_2|^2$.

(ii) *Inside the hyperbolae*: In this case, $\mathcal{K}$ is a decreasing function of both $|\delta_1|$ and $|\delta_2|$, attaining its maximum at $\min |\delta_1|, |\delta_2|$. Given that $\delta_1, \delta_2 \neq 0 \Rightarrow |\delta_1|^2, |\delta_2|^2 \geq d_{min}^2$, one gets the minimum value

$$\mathcal{K}\Big|_{|\delta_1|^2 = |\delta_2|^2 = d_{min}^2} = -\frac{1}{2}.$$

Comparing (i) and (ii) yields, subject to $\delta_n \neq 0$,

$$\max_{\boldsymbol{\delta}} \mathcal{K} = -\frac{1}{2}.$$

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Lemma 3: Assuming $\Delta fT \in \mathbb{Q}$,

$$\min_k \cos[\theta_1 - \theta_2 + \pi(\Delta fT + \epsilon)(2k-1)] = -1, \tag{11}$$

for some $\epsilon \to 0$.

Proof. This proof is conducted by constructing $\epsilon \to 0$ for which (11) holds. For simplicity, the derivations are separated in two cases:

(i) $\Delta fT = \frac{P}{O}$, *$P, O \in \mathbb{Z}$ and O an odd number*: Choose

$$\epsilon = -\frac{\beta\pi + \theta_1 - \theta_2}{(2\alpha+1)O\pi}, \; \alpha \in \mathbb{Z},$$

with $\beta = 1$ if P is even and $\beta = 0$ otherwise. For $2k-1 = (2\alpha+1)O$, (11) simplifies to $\cos[(2\alpha+1)N\pi + \beta\pi] = -1$. Notice that $\lim_{\alpha\to\infty} \epsilon = 0$.

(ii) $\Delta fT = \frac{P}{E}$, *$P, E \in \mathbb{Z}$ and E an even number*: Choose

$$\epsilon = \frac{N}{E(E\alpha - 1)} - \frac{\beta\pi + \theta_1 - \theta_2}{\pi(E\alpha - 1)}, \; \alpha \in \mathbb{Z},$$

again with $\beta = 1$ if P is even and $\beta = 0$ otherwise. In this case, (11) becomes

$$\cos\left[\theta_1 - \theta_2 + \pi(2k-1)\left(\frac{\alpha N}{E\alpha - 1} - \frac{\beta\pi + \theta_1 - \theta_2}{\pi(E\alpha - 1)}\right)\right],$$

where $E\alpha - 1$ is odd. Hence, for $2k-1 = E\alpha - 1$, (11) yields $\cos(\alpha N\pi + \beta\pi) = -1$. Again, $\lim_{\alpha\to\infty} \epsilon = 0$.

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Using the previous results, Theorem 1 is proved as follows.

Proof of Theorem 1. From Lemma 1, the equality in (6) holds for all ΔfT whenever $\delta_1 = 0$ or $\delta_2 = 0$.

For $\delta_n \neq 0$, $n = 1, 2$, condition (6) reduces to (9). Given that $\cos(x) \geq -1$, it is necessary that

$$\frac{\mathcal{K}}{\operatorname{sinc}(\Delta fT)} \leq -1$$

for (9) to hold. Moreover, the lower bound on ΔfT will occur for the maximum value of $\mathcal{K}$ given that sinc is a decreasing function in $[0, 1]$. Hence, from Lemma 2, $\operatorname{sinc}(\Delta fT) \leq 0.5$. The value of ΔfT can be approximated using the Taylor series expansion of the sinc function yielding $\Delta fT > 0.6033$.

Notice, however, that this condition is necessary ($\Leftarrow$) for (8) but not sufficient ($\Rightarrow$), since the cosine on the left-hand side of (9) can be strictly larger than -1 for the value of ΔfT found above. Nevertheless, Lemma 3 guarantees that infinitesimally close to any ΔfT there exists a $\Delta fT'$ for which $\min_k \cos[\theta_1 - \theta_2 + \phi(k)] = -1$. Since both cos and sinc are smooth function [11]—in the sense that they have derivatives of all orders—, (8) is an infinitesimally tight bound for sufficiency.

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IV. Simulations

This section starts with simulations to illustrate Theorem 1. So as to show the validity of the derivations for any $\mathcal{C}$, the results are presented for a square 16-QAM constellation, a hexagonal 16-QAM constellation, and a constellation composed of 8 randomly chosen symbols (Fig. 2). In Fig. 3, the minimum distance between the transmitted multicarrier symbols—relative to the minimum distance in the respective constellations, d_{min}^2—is presented for $N = 2$ and different ΔfT. Notice that there is a clear threshold above which

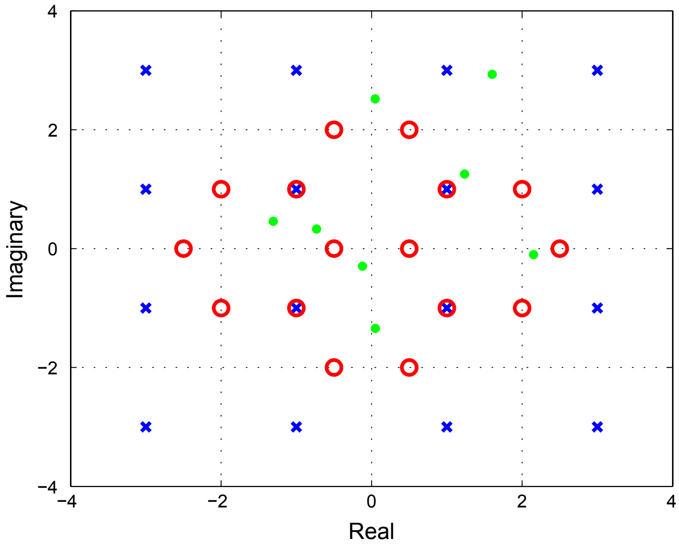

Fig. 2. Constellations: × – Square ($M = 16$); O – Hexagonal ($M = 16$); • – Random ($M = 8$)

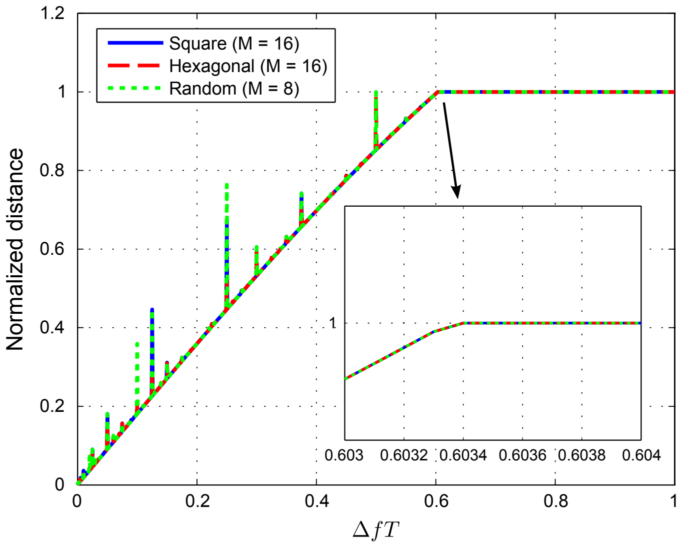

Fig. 3. Minimum relative distance for different ΔfT

$\min D_{ij}^2 = d_{min}^2$. The detail in Fig. 3 confirms that this threshold is the one provided by Theorem 1. Similar figures can be found in [6], [7], [9] for square 4-QAM and 36-QAM constellations.

Notice that below the deduced lower bound the minimum distance decreases with the relative spectral separation. This decay, however, is not monotonic. This is a result of the fact that for some ΔfT, there exist directions—$(\theta_1 - \theta_2)$—for which the cosine in (9) is strictly larger that -1. Thus, non-OFDM has preferential orientations in the signal space that can be exploited to further improve spectral efficiency.

Finally, Fig. 4 shows the BER as a function of SNR for a 4-QAM signal transmitted over two carriers with different spectral separations. Notice that for a wide range of ΔfT the non-orthogonal communication scheme performs as well as an OFDM, but with higher spectral efficiency—up to almost 40%. Only when ΔfT drops below the spectral separation bound from Theorem 1 does the BER start to increase.

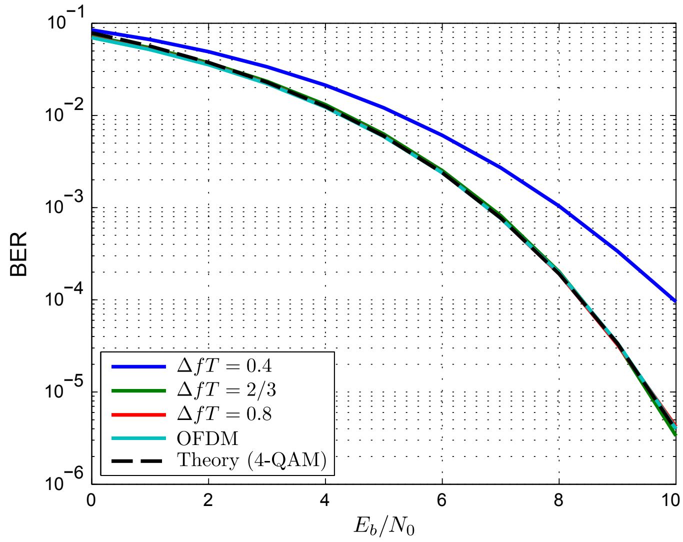

Fig. 4. BER of 4-QAM with square constellation for different ΔfT

V. Conclusions

This work derived the lower bound for spectral separation above which the BER of a non-OFDM system with two carriers is the same as that of an OFDM. This analytical bound is valid for any complex constellation and shows that non-OFDM communication can be almost 40% more spectrally efficient than OFDM with no change in the probability of error. Although this phenomenon was observed empirically in [6]–[8], it remained an open theoretical issue. Future development of this work include the analysis of non-OFDM systems with an arbitrary number of carriers, the design of constellations that further improve spectral efficiency, and the development of equalization and demodulation techniques.

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