Eigenmodes of a Toroidal Cavity with a Conducting Separating Wall

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Abstract—The eigenmodes of a toroidal cavity with a separating wall are examined in the light of theory and experiment. Excellent agreement between the predicted resonant frequency of 1.148 GHz corresponding to a 4π-period generating function and the experimentally observed frequency is found. The exact solution of Maxwell's equations in toroidal systems with a separating wall is presented and the eigenfrequencies are experimentally measured on a toroidal cavity with major and minor radii of respectively 12.5 cm and 10.0 cm.

Index Terms—toroidal resonators, microwave field theory and plasma heating.

I. INTRODUCTION

The motivation of the present work stems from an alternate approach for heating toroidal plasmas by RF waves. The main idea is to treat the toroidal vessel as a resonant cavity and to look at global wave solutions. The plasma-filled cavity would then absorb resonantly the energy of electromagnetic waves at proper frequencies launched into the system.

As long as the wavelength is smaller than the torus radii, geometric-optic ray tracing methods are suitable to describe the plasma-wave coupling mechanism. If the wavelength is, however, comparable to the length scale of the system, then propagation, resonance and eigenfrequencies of the electromagnetic waves should be determined by solving Maxwell's equations in toroidal coordinates. To achieve this goal, it is important that the vacuum toroidal cavity should be characterized first.

Early experimental characterization studies of a toroidal cavity having a circular cross section has begun since the seventies [1]. On the hand, eigenfrequency calculations and identification of normal modes have already been addressed by many authors, yet such a theoretical problem still remains an open question since all the solutions so far obtained are approximate in nature [2]-[9]. The fundamental crux is that the toroidal geometry permits no strictly TM or TE separable eigenmodes as in the case of cylindrical coordinates. Nevertheless, the exact solution of Maxwell's equations for electromagnetic waves in toroidal resonators with a separating wall was obtained in 1979 by Deutsch [10].

In the present work we endeavor to verify experimentally the exact solution indicated in [10], as well as to give insights into the toroidal resonator problem. We should mention that the results presented here are preliminary ones, that will be supplemented and largely extended in forthcoming studies.

In addition to having practical use in optical fibers and storage rings, toroidal hollow waveguides find applications as antenna feeds or as closed resonators in particle physics. Further inclusion of a magnetized plasma into the problem could contribute for implementing an RF heating system on the Spherical Tokamak Experiment (ETE) which is currently in final assembly stage at LAP/INPE [11].

Following the Deutsch's approach, we describe in the next Section the electromagnetic field in the toroidal resonator with a separating wall and the corresponding exact generating function with periodicity 4π, which is compared to the experimental results given in Section III. Discussion and conclusions of our work are contained in Sections IV and V, respectively.

II. THEORETICAL APPROACH

This theoretical approach is a condensation from [7] and [10]. The coordinate system used to describe the problem is the usual r, θ, ϕ local toroidal coordinate system, defined by:

$$x = (R_0 + r\ cos\theta)\ cos\phi \tag{1}$$

$$y = (R_0 + r\ cos\theta)\ sen\phi \tag{2}$$

$$z = r\ sen\theta \tag{3}$$

The study of the toroidal resonator for electromagnetic waves is based on the vector Helmholtz equation for the Hertz vector. In cartesian coordinates the vector equation splits into the system of three equations:

$$\vec{\nabla}^2\Pi_j + \mu_0\epsilon_0\omega^2\Pi_j = 0 \tag{4}$$

where $j = x$, y and z.

The electric $\vec{E}$ and magnetic $\vec{B}$ fields are recovered

by:

$$\vec{E} = \vec{\nabla}(\vec{\nabla}.\vec{\Pi}) - \mu_0\epsilon_0\frac{\partial^2\vec{\Pi}}{\partial t^2} \quad (5)$$

and

$$\vec{B} = \mu_0\epsilon_0\vec{\nabla} \times \frac{\partial\vec{\Pi}}{\partial t} \quad (6)$$

Equation (4) is transformed to local toroidal coordinates and the components Π_j are regarded as function of r, θ and ϕ. Then, a scalar Helmholtz equation is obtained:

$$\frac{\partial}{\partial\rho}\,\rho\,(1-\rho\,cos\theta)\,\frac{\partial\Pi_j}{\partial\rho} + \frac{\partial}{\partial\theta}\,\frac{(1-\rho\,cos\theta)}{\rho}\,\frac{\partial\Pi_j}{\partial\theta}$$

$$+\frac{\rho}{(1-\rho\,cos\theta)}\,\frac{\partial^2\Pi_j}{\partial\phi^2} + k^2\rho\,(1-\rho\,cos\theta)\,\Pi_j = 0 \quad (7)$$

where $k^2 = \mu_0\,\epsilon_0\,\omega^2\,R_0^2$ and $\rho = r/R_0$. Here R_0 is the major torus radius.

In order to solve this equation the following substitution is made:

$$\Pi_j(\rho,\theta,\phi) = \frac{\psi_m(\rho,\theta)\;e^{im\varphi}}{\sqrt{1-\rho\,cos\theta}} \quad (8)$$

Inserting into (7) we get:

$$\frac{\partial^2\psi_m}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\psi_m}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2\psi_m}{\partial\theta^2} + k^2\psi_m = \frac{m^2-1/4}{(1-\rho\,cos\theta)^2}\,\psi_m \quad (9)$$

Equation (9) is not separable, but the expression from the right-hand side vanishes if $m = 1/2$ and the choice of this m value goes to an exact solution of the equation. This solution has a period of 4π and correspond to the toroidal resonator with a conducting separating wall [10].

As is shown in [7] and [10], three different toroidal eigenmodes can be defined through the cartesian components of the Hertz vector. In this paper, we shall look into two types of normal modes:

$$\vec{\Pi} = (f\,sin\,\theta, f\,cos\,\theta, 0) \quad (10)$$

$$\vec{\Pi} = (-f\,cos\,\theta\,cos\,\phi, f\,sin\,\theta\,cos\,\phi, -f sin\,\phi) \quad (11)$$

The third normal mode differs from (11) only by a rotation through $\pi/2$ around the axis of the torus.

The function f that appears in (10) and (11) is the exact solution above mentioned. It is given by:

$$f\,(\rho,\theta,\phi) = \frac{J_\nu(k\,\rho)}{\sqrt{1-\rho\,cos\theta}}\;cos(\nu\,\theta-\alpha)\,sin\frac{\phi}{2} \quad (12)$$

where the $J_\nu(k\,\rho)$ is the Bessel function and ν is the θ variable separating constant.

At the toroidal surface, the normal component of the magnetic field must vanish. This is achieved if:

$$J_\nu(k\,\rho_0) = 0 \quad (13)$$

where $\rho_0 = a/R_0$ is the inverse aspect ratio of the torus and a is the minor radius. Simultaneously, the ϕ-component of $\vec{B}$ must vanish at the conducting separating wall, but this condition is automatically fulfilled for the plane $\phi = 0$ by the f function, and so, the condition (13) leads to:

$$k\,\rho_0 = j_{\nu,1} \quad (14)$$

where $j_{\nu,1}$ is the first root of the J_ν. Using (14), and remembering the definitions of k and ρ_0, we get the dispersion relation:

$$\omega_\nu = \frac{c}{a}\,j_{\nu,1} \quad (15)$$

where c is the light speed. Note that the resonator eigenfrequencies depend only on the minor torus radius. The values calculated from (15), will be compared with the experimental data given in the next Section.

III. EXPERIMENTAL ARRANGEMENT

The experimental arrangement to verify the eigenfrequency given by (15) consists of a network analyzer (HP 8757D) and a copper toroidal cavity with major and minor radii $R_0 = 12.5cm$ and $a = 10.0cm$, respectively. The analyzer has three interconnected modules, namely, *i*) a sweep generator – that generates a high-frequency signal, *ii*) an analyzer module – that processes the cavity signal response, and *iii*) a detecting unit – that detects the resonant electromagnetic fields. The instrument being fully digital, the chosen frequency range to be covered is sampled using a given number of discrete points, and in our experimental measurements 401 sampling points have been selected.

To probe the polarization of the resonant fields, the resonator excitation has been accomplished with a magnetic loop oriented in two ways. In the first, the loop is oriented in a plane perpendicular to the toroidal axis, and in the second orientation the loop plane is made normal to the resonator's cross section.

Detection of resonant modes has been carried out via an electric probe connected to the detecting unit of the analyzer. The exciting and detecting probes have been introduced in the resonator through the top surface and placed 180^0 degrees apart, and, as such, the transmission method has been adopted to identify the resonant frequencies.

Two sets of data have been taken: the first set refers to the hollow resonator, while the second data set has been obtained with a 2mm-thick copper disk inserted into the torus and exactly fitting its circular cross section. The experiment has been made on the 1.110 - 1.165 GHz range, and, according to (15), the first two eigenfrequencies associated with the 4π-period generating function are 1.148 GHz and 1.830 GHz.

In Figs. 1(a) and 2(a) we show the mode spectra for the hollow and disk-loaded cavities, respectively, in the case of the exciting magnetic loop being perpendicular to the toroidal axis; similarly, Figs. 1(b)

and 2(b) give the corresponding spectra with the loop normal to the cavity cross section.

IV. DISCUSSION

Comparison between theoretical and numerical findings relies on three considerations: the eigenfrequency predicted by (15), the electromagnetic field pattern, and the Slater perturbation measurement method [12]. On superimposing Fig. 1(a) on Fig. 2(a) or Fig. 1(b) on Fig. 2(b), a frequency shift between the spectra is cleary noticed, such that the resonance peaks corresponding to the hollow cavity occur at lower frequencies. However, two peaks remain almost coincident irrespective of the orientation of the exciting loop. By taking the average of the values provided by the four plots, we can infer that the coinciding peaks are centered at 1.12578 GHz and 1.14149 GHz. The percent difference between the higher resonant frequency and that predicted by (15) is as small as 0.6%, which strongly supports that the observed value of 1.14149 GHz is indeed the eigenfrequency associated with the exact solution with the periodicity of 4π. The lower frequency value has no correspondence with any eigenvalue as indicated by (15).

To ascertain the frequencies associated with the coinciding peaks we shall look into the field patterns and comment upon basic ideas of the Slater perturbation method [12]. This method had been originally developed for measuring the electric field axial profile in accelerator cavities, and later on it was extended to other microwave devices. It is based on the fact that the presence of a perturbing object gives rise to a shift in the natural resonant frequency by an amount which depends upon the local intensity of the electromagnetic field where the object is located.

Mathematically, this statement is expressed by [13]:

$$\frac{\Delta\omega}{\omega_m} = \frac{\int_v [\vec{E}_1\vec{D}_0 - \vec{E}_0\vec{D}_1 + \vec{H}_0\vec{B}_1 - \vec{H}_1\vec{B}_0]\,dv}{\int_V [\vec{E}_0\vec{D}_0 - \vec{B}_0\vec{H}_0]\,dV} \qquad (16)$$

where $\Delta\omega$ is the frequency shift, ω_m is the unperturbed resonant frequency, V is the volume of the cavity, v is the volume of the perturbation probe and the subscripts 1 and 0 indicate perturbated and nonperturbated values of the electromagnetic fields.

It is possible, however, that the perturbing object does not entail any change in the resonant frequency. To this end, two possibilities are likely achievable. In the first alternative the object is placed where both the electric and magnetic field intensities are vanishingly small. The second possibility ties the local electromagnetic field pattern in with the shape of the perturbing volume. Let us discuss our experimental results in the light of the second alternative.

Inasmuch as two resonant frequencies remain unchanged upon introducing the separating disk, we can reason that there both exist inside the torus a circunferential electric field line (thus normal to the disk) and a magnetic surface whose magnetic field lines are tangential to the disk. This latter boundary condition that the toroidal ϕ-component of the magnetic field must vanish at the disk surface is automatically fulfilled by the generating function (12) for the plane $\phi = 0$. We conclude, therefore, that the detected frequency of 1.14149 GHz is indeed the eigenfrequency predicted by the exact solution (12), while the frequency of 1.12578 GHz should correspond to a eigenmode having a toroidally simetric electric field. Moreover, the dense spectrum with many resonant frequencies that have been detected with or without the presence of the disk is indicative of the complex structure of the electromagnetic field in toroidal systems [2]-[10]. And the fact that the cavity has been excited via a magnetic loop and the fields detected with an electric probe is an experimental demonstration that the toroidal cavity fields are neither of the TE nor TM types [2]-[10].

V. CONCLUSION

An experiment has been conducted to verify the exact solution of Maxwell's equations for a torus with a separating wall. To our knowledge this is the first time that close agreement between theory and experiment concerning a toroidal cavity is found.

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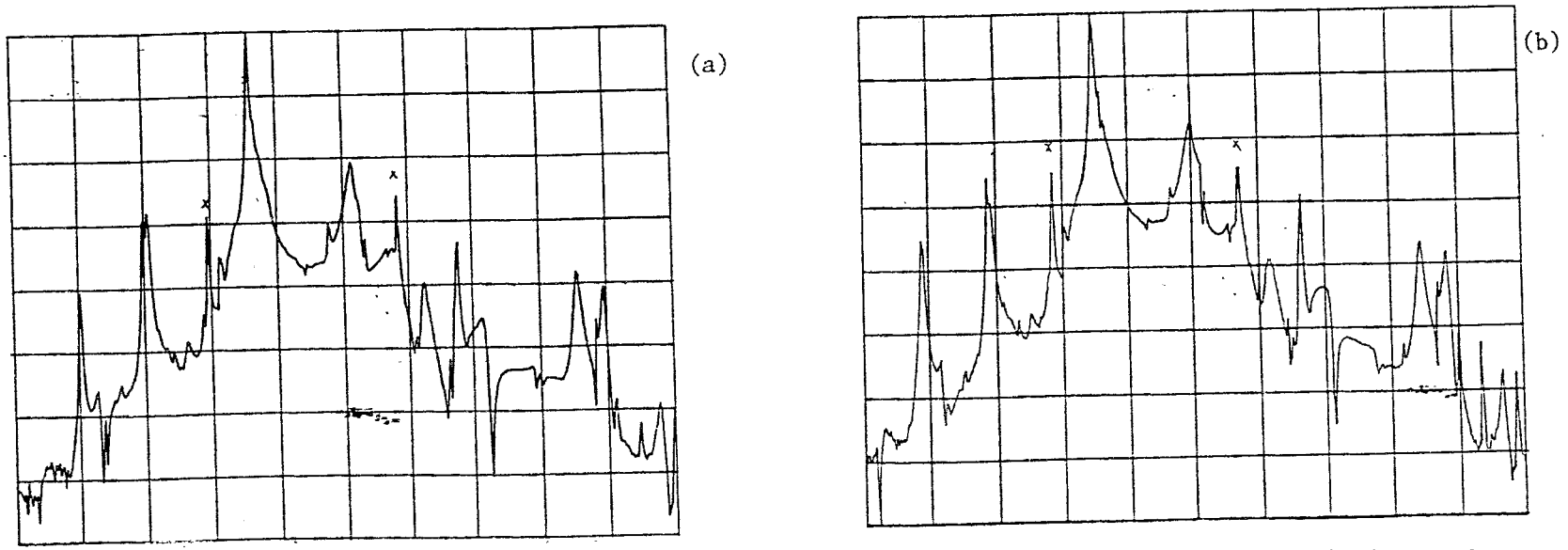


Fig. 1 Mode spectra of the hollow toroidal cavity excited via a magnetic loop normal to (a) the toroidal axis and to (b) the torus' cross section

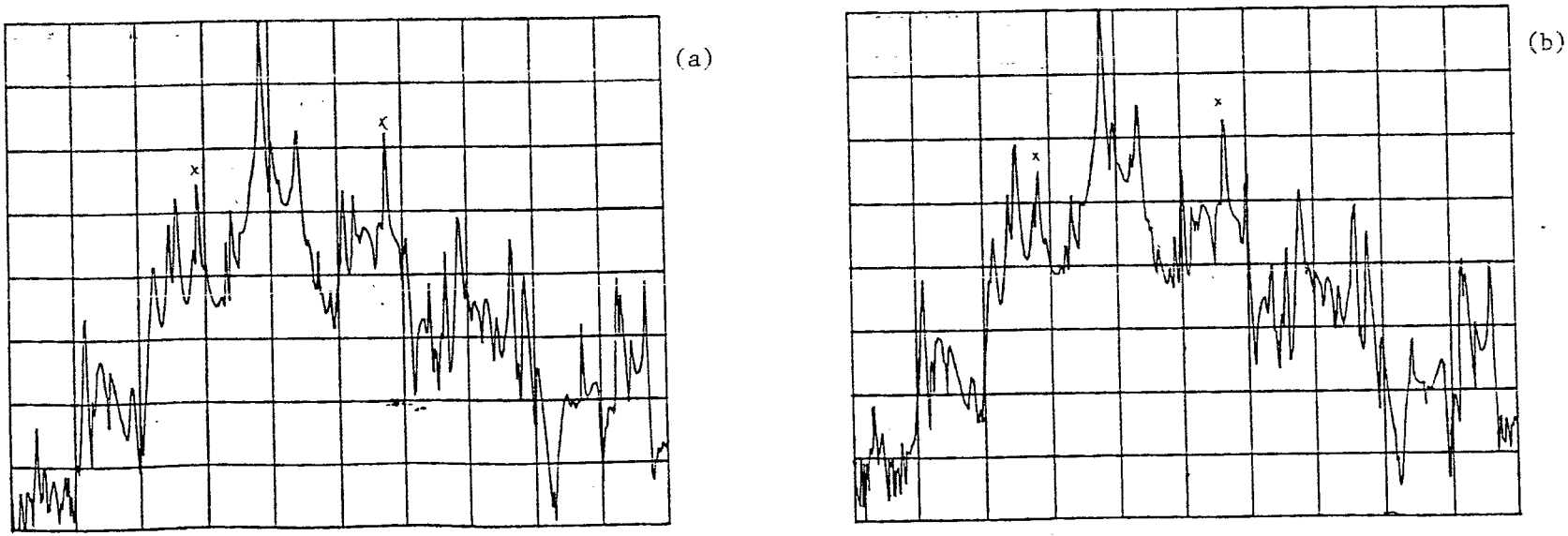


Fig. 2 The same as above but with the separating metallic disk