INPE – National Institute for Space Research
São José dos Campos – SP – Brazil – May 16-20, 2016

SYNCHRONIZATION OF PHASE OSCILLATORS WITH COUPLING MEDIATED BY A DIFFUSING SUBSTANCE

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Abstract: We investigate the synchronization of pointlike phase oscillators ("cells") whose coupling is mediated by a chemical diffusing in the intercell medium. We consider how frequency synchronization is affected by the non-local features of the coupling prescription.

keywords: Synchronization in Nonlinear Systems, Nonlinear Dynamics and Complex Systems, Complex Networks.

1. INTRODUCTION

Networks of coupled phase oscillators have been extensively used in many physical and biological applications [1]. A number of different coupling prescriptions has been studied, but most attention has been given to two types: (i) local (or Laplacian) coupling, where each oscillator interacts with its nearest neighbors; and (ii) global coupling, where an oscillator is coupled to the mean field of all other oscillators regardless of their position.

However, in many applications it turns out that an intermediate form of coupling would better describe the system, since each oscillator interacts with its neighbors but with a strength which depends on the mutual spatial distance. Such non-local couplings have been studied for a long time but recently it was recognized their importance in the production of chimera states.

One of the interesting dynamical aspects to be studied in networks of non-locally coupled phase oscillators is synchronization, which is a universal phenomenon [2]. A paradigmatic example of them is the synchronization of flashing fireflies, which interact by the emission of light pulses. Since the velocity of light is very large, the coupling effect is rapidly spread along the network and fireflies can flash in unison, producing an impressive phenomenon.

On the other hand, emission of light pulses is not the only way oscillators can use to communicate among themselves. Another possibility is the emission and absorption of a chemical substance which diffuses in the medium contaning the oscillators. A theory for describing such phenomena has been proposed by Kuramoto, and in the case of fast relaxation it amounts to an interaction which decays exponentially with the distance between oscillators [3–6].

In this work we consider a model of pointlike phase oscillators which interact according to this non-local coupling prescription. We investigate in what extent the frequency synchronization properties vary with the coupling parameters. It turns out that there is a synchronization transition with properties depending on both the coupling strength and range.

2. MODEL

We use two classes of vectors: (i) positions $\vec{r}$ in a d-dimensional Euclidean space; (ii) state variables $\mathbf{X} =$

$(x_1, x_2, \ldots x_M)^T$ in a M-dimensional phase space of the dynamical variables characterizing the state of the system at a given time t. There are N oscillator cells located at discrete positions $\vec{r}_j$, where $j = 1, 2, \cdots N$, in the d-dimensional Euclidean space; and $\mathbf{X}_j$ is the state variable for each oscillator, whose time evolution is governed by the vector field $\mathbf{F}(\mathbf{X}_j)$. The time evolution of the state variable is affected by the local concentration of a chemical, denoted as $A(\vec{r}, t)$, through a time-dependent coupling function $\mathbf{g}$:

$$\frac{d\mathbf{X}_j}{dt} = \mathbf{F}(\mathbf{X}_j) + \mathbf{g}(A(\vec{r}_j, t)), \tag{1}$$

and the chemical concentration satisfies a diffusion equation

$$\varepsilon \frac{\partial A}{\partial t} = -\eta A + D\nabla^2 A + \sum_{k=1}^{N} h(\mathbf{X}_k)\delta(\vec{r} - \vec{r}_k), \tag{2}$$

where $\varepsilon \ll 1$ is a small parameter representing the fact that diffusion occurs in a timescale faster than the intrinsic period of individual oscillators; η is a phenomenological damping parameter (representing the chemical degradation of the mediating substance), and D is a diffusion coefficient. The diffusion equation above has a source term h which depends on the oscillator state at the discrete positions $\vec{r}_j$: this means that each oscillator secrets the chemical with a rate depending on the current value of its own state variable.

If the diffusion is much faster than the oscillator period, we can assume $\varepsilon \dot{A} = 0$ such that the concentration relaxes to a stationary value

$$A(\vec{r}_j) = \sum_{k=1}^{N} \sigma(\vec{r}_j - \vec{r}_k) h(\mathbf{X}_k), \tag{3}$$

where σ is a Green function, obtained by solving

$$-D\nabla^2 \sigma + \eta\sigma = \delta(\vec{r}_j). \tag{4}$$

Substituting (3) into (2) we obtain

$$\frac{d\mathbf{X}_j}{dt} = \mathbf{F}(\mathbf{X}_j) + \mathbf{g}\left(\sum_{k=1}^{N} \sigma(\vec{r}_j - \vec{r}_k) h(\mathbf{X}_k)\right). \tag{5}$$

We adopt the following form of the coupling function

$$\mathbf{g}(h(\mathbf{X}_k)) = \mathbf{A}\mathbf{H}(\mathbf{X}_k), \tag{6}$$

where $\mathbf{A}$ is a $M \times M$ matrix indicating which variables of the oscillators are coupled to whom, and $\mathbf{H}$ is a nonlinear function of its arguments. Inspired by Kuramoto model of coupled phase oscillators [7] we set $M = 1$ such that $\mathbf{X}_j$ is a phase $\theta_j \in [0, 2\pi)$, and the vector function $\mathbf{F}(\mathbf{X}_j)$ is the corresponding frequency ω_j. In this case $\mathbf{A}$ reduces to a scalar coupling strength K and the nonlinear coupling function is sinusoidal, which results in

$$\dot{\theta}_j = \omega_j + K \sum_{k=1}^{N} \sigma(\vec{r}_j - \vec{r}_k) \sin(\theta_k - \theta_j). \tag{7}$$

A solution of Eq. (4), in the one-dimensional case, is $\sigma(r) = C\exp(-\gamma r)$, where $\gamma = \sqrt{\eta/D}$ and C is determined from a normalization condition. In a one-dimensional lattice (7) can be written in the following form

$$\dot{\theta}_j = \omega_j + \frac{K}{\alpha} \sum_{\ell=1}^{N'} e^{-\gamma\ell} \left[\sin(\theta_{j-\ell} - \theta_j) + \sin(\theta_{j+\ell} - \theta_j)\right], \tag{8}$$

where $\alpha = 2\sum_{\ell=1}^{N'} e^{-\gamma\ell}$ and $j = 1, 2, \ldots N$.

In the numerical simulations to be shown in this work we integrated the above system of differential equations using a fourth-order Runge-Kutta method with variable stepsize. The initial conditions $\theta_j(t = 0)$ were chosen within the interval $[0, 2\pi)$ using a uniform probability distribution. We used periodic boundary conditions: $\theta_{j\pm N} = \theta_j$.

3. SYNCHRONIZATION OF FREQUENCIES

The frequency of the coupled oscillators is defined as

$$\Omega_j(t) = \lim_{t\to\infty} \frac{\theta_j(t+T) - \theta_j(T)}{t}, \qquad (j = 1, 2, \ldots N), \tag{9}$$

where T is chosen such that transients have decayed. Two or more adjacent oscillators are synchronized if this frequency is equal, up to some tolerance. The non-locality of the coupling causes the formation of synchronization with different lengths. If N_i is the length of the ith plateau, and N_p the total number of them, with average size $\langle N\rangle = (1/N_p)\sum_{i=1}^{N_p} N_i$, the synchronization degree P is the ratio between the average plateau length and the total lattice size: $P = \langle N\rangle/N$.

For a totally synchronized state we have one plateau comprising the entire lattice ($\langle N\rangle = N$) so that $P = 1$. On the other hand, for a completely non-synchronized state $N_p \approx N$, or $\langle N\rangle \approx 1$, giving $p \approx 1/N \to 0$, if $N \to \infty$. In Fig. 1 we plot the variation of the synchronization degree with the parameter γ for a fixed coupling strength K and different lattice sizes. If γ is small enough the oscillators are frequency-synchronized and, as γ increases past a critical value γ_c, there is a transition to a non-synchronized state (or, more properly, to a partially synchronized one). It turns out that γ_c decreases as we increase the lattice size.

If we set a fixed value of γ and vary the coupling strength K, a similar transition to frequency synchronization occurs, after K increases past a critical value K_c. As a matter of fact, the occurrence of frequency synchronization depends on both coupling parameters (γ and K) as illustrated by Fig. 2, where we depict the region of frequency synchronization (according to the value of the synchronization degree P) as a function of γ and K. The transition to frequency synchronization occurs past a critical line. The critical coupling strength in the globally coupled case is $K_c \approx 0.08$, as can be seen in the inset of Fig. 2.

4. DISCUSSION

The numerical results presented in the previous section can be interpreted in the light of the competition between

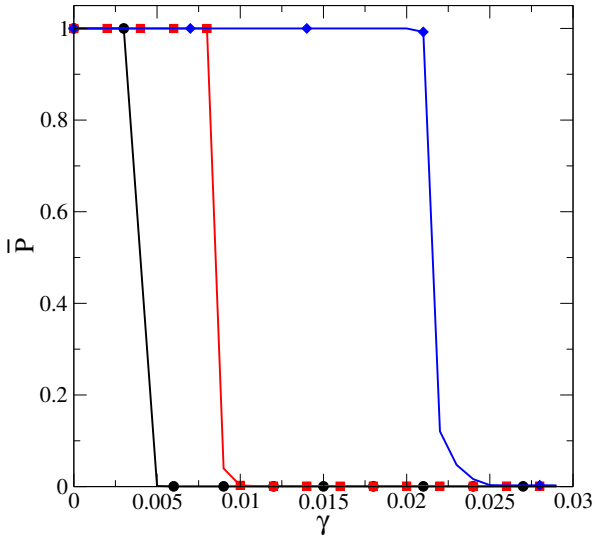

Figure 1 – Frequency synchronization degree as a function of the inverse coupling length for $K = 7$ and different lattice sizes: $N = 401$ (blue), 1001 (red), and 2001 (black).

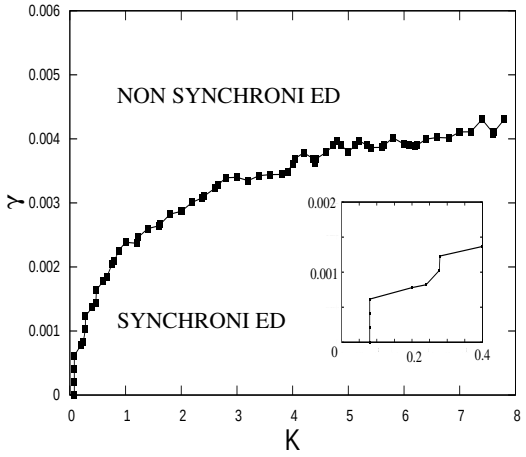

Figure 2 – Regions of frequency synchronization, according to the value of the synchronization degree P, *vs.* coupling parameters (coupling strength K and inverse coupling length γ) for a chain of $N = 2001$ Kuramoto oscillators. The squares indicate the boundary of the frequency-synchronized region. In the inset we show a magnification of the lower right corner of the figure.

coupling range and the disorder characteristic of the randomness of the frequencies (chosen from a Gaussian frequency distribution). In order to do so, let us first consider the limiting cases of the nonlocal form of coupling we used in this work. If $\gamma \to 0$ then $C = 1/N - 1$ and we have a global type of coupling

$$\dot{\theta}_j = \omega_j + \frac{K}{N-1} \sum_{k=1}^{N} \sin\left(\theta_k - \theta_j\right), \qquad (10)$$

which is very similar to the classical Kuramoto model of coupled phase oscillators [7].

In the case of large γ, the coupling decays very fast with the lattice distance ℓ, such that only the term with $\ell = 1$ contributes significantly to the summations, yielding $C \approx e^{\gamma\Delta}/2$ and the coupling takes into account only the nearest neighbors of a given site

$$\dot{\theta}_j = \omega_j + \frac{K}{2}\left[\sin\left(\theta_{j-1} - \theta_j\right) + \sin\left(\theta_{j+1} - \theta_j\right)\right] \qquad (11)$$

as is the usual local (diffusive or Laplacian) coupling.

Since uncoupled oscillators have randomly distributed natural frequencies, the onset of frequency synchronization appears due to the dominant effect of the coupling over the randomness of the original distribution. For global couplings ($\gamma = 0$) the value of K_c is close to zero, indicating that a synchronized state occurs for very weak couplings. Local couplings, where γ is large enough, however, may not yield frequency synchronization even if K is very large.

5. CONCLUSIONS

The frequency synchronization properties of a network of nonlocally coupled oscillators are strongly dependent on both the coupling strength and range. The latter makes is possible to go from the global (all-to-all) to local (nearest-neighbor) coupling by varying a single parameter. We showed that there is a synchronization transition as the coupling strength is varied through a critical value. In the global case this value can be described by a mean-field theory but, as the range parameter increases the critical value of K also increases. Hence, for locally coupled lattices, although the synchronization transition could be possible in principle, the critical coupling strength would be very large.

ACKNOWLEDGEMENTS

This work was made possible with the help of CNPq, CAPES, FAPESP, and Fundação Araucária (Brazilian Government Agencies).

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