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Key Points:

- Different assumptions about the interactions between air and condensate
- Different equations of motion during evaporation and condensation
- Resolving contrasting formulations in the literature

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The equations of motion for moist atmospheric air

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Abstract How phase transitions affect the motion of moist atmospheric air remains controversial. In the early 2000s two distinct differential equations of motion were proposed. Besides their contrasting formulations for the acceleration of condensate, the equations differ concerning the presence/absence of a term equal to the rate of phase transitions multiplied by the difference in velocity between condensate and air. This term was interpreted in the literature as the "reactive motion" associated with condensation. The reasoning behind this reactive motion was that when water vapor condenses and droplets begin to fall the remaining gas must move upward to conserve momentum. Here we show that the two contrasting formulations imply distinct assumptions about how gaseous air and condensate particles interact. We show that these assumptions cannot be simultaneously applicable to condensation and evaporation. Reactive motion leading to an upward acceleration of air during condensation does not exist. The reactive motion term can be justified for evaporation only; it describes the downward acceleration of air. We emphasize the difference between the equations of motion (i.e., equations constraining velocity) and those constraining momentum (i.e., equations of motion and continuity combined). We show that owing to the imprecise nature of the continuity equations, consideration of total momentum can be misleading and that this led to the reactive motion controversy. Finally, we provide a revised and generally applicable equation for the motion of moist air.

1. Introduction

The equation of motion for moist air in the presence of phase changes remains controversial in the meteorological and multiphase flow literature [*Young*, 1995; *Drew and Passman*, 1998; *Ooyama*, 2001; *Bannon*, 2002; *Brennen*, 2005]. *Young* [1995] for example reviewed this subject and highlighted a number of inconsistencies among published treatments. In the atmospheric sciences this problem received attention in the works of *Ooyama* [2001] and *Bannon* [2002] (hereafter O01 and B02, respectively). But rather than resolving the issues these authors offered contrasting equations derived from first principles. *Cotton et al.* [2011] reviewed the fundamental equations of moist atmospheric dynamics and noted that the correct way to consider phase changes remained poorly understood.

Here we consider the differences between the formulations of O01 and B02 summarized in Figure 1, equations (1)–(14). For clarity, and without losing generality, this summary considers the case when there is only one type of condensate particles (droplets with velocity $\mathbf{v}_c$) and the external forces acting on moist air are gravity and pressure gradient (Coriolis force and friction are not considered). The two formulations for moist air, equations (13) and (14), differ in that B02 includes the following term

$$\mathbf{W}\dot{\rho}_a, \quad \mathbf{W} \equiv \mathbf{v}_c - \mathbf{v}_a, \tag{15}$$

where $\mathbf{W}$ is the difference between condensate velocity $\mathbf{v}_c$ and air velocity $\mathbf{v}_a$, $\dot{\rho}_a$ is the rate of phase transitions (kg m^{-3} s^{-1}). When condensation occurs, $\dot{\rho}_a < 0$, see equation (1) in Figure 1. (The additional differences between equations (13) and (14) will be addressed later.)

Bannon [2002, p. 1972] interpreted this term as the "reactive motion" arising during condensation: as the droplets begin to fall and thus gain a downward velocity, the remaining air gains an upward velocity so that

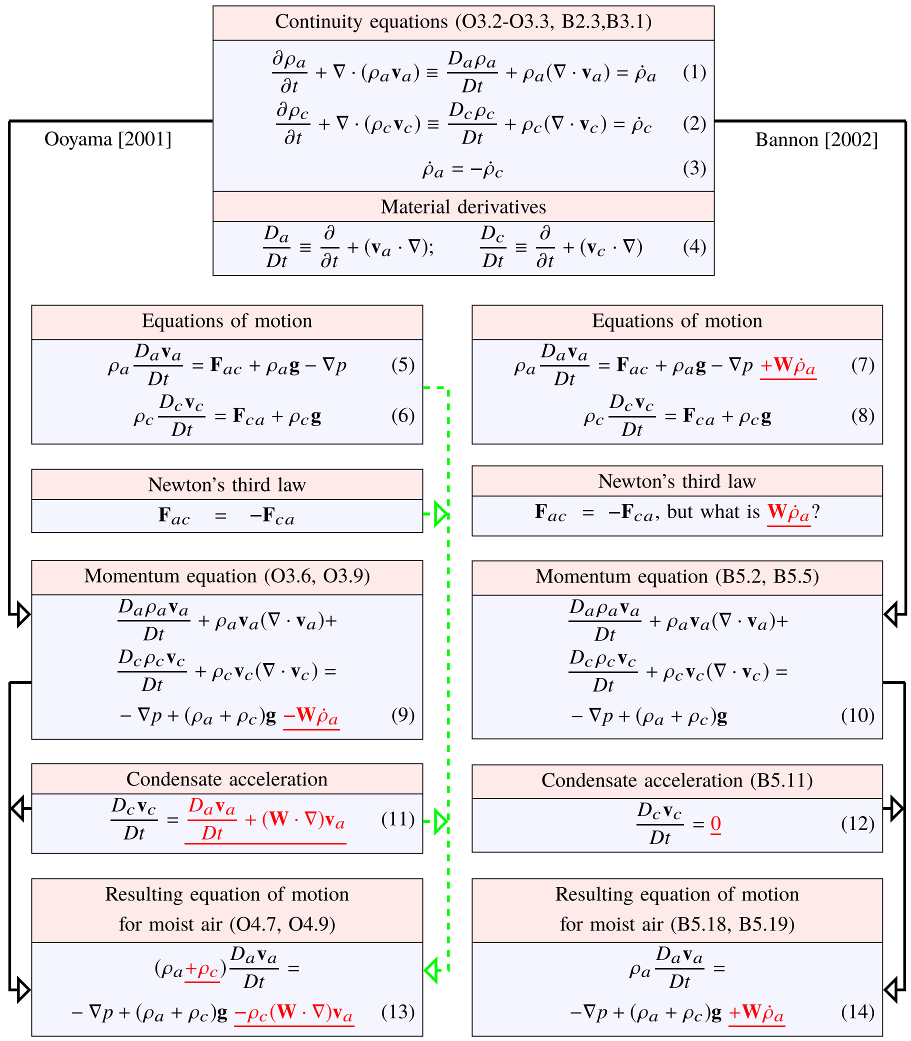

Figure 1. Key relationships of (left) O01 and (right) B02 with differences underlined and highlighted in red. Subscripts a and c refer to moist air (dry air and water vapor) and condensate, respectively; ρ (kg m^{-3}) is density; $\dot{\rho}$ (kg m^{-3} s^{-1}) is the rate of phase transitions, $\mathbf{v}$ is velocity; $\mathbf{W} \equiv \mathbf{v}_c - \mathbf{v}_a$; p is pressure, $\mathbf{g}$ is the acceleration of gravity. Black solid arrows indicate how the equation of motion for moist air was derived from the momentum equation: multiply the continuity equations for air and condensate by, respectively, $\mathbf{v}_a$ and $\mathbf{v}_c$, subtract them from the momentum equation, and use the assumption about condensate acceleration. Green dashed arrows indicate the derivation from the equations of motion and Newton's third law (not applicable for B02). Numbers in parentheses marked with "O" and "B" refer to the corresponding equations of O01 and B02.

the combined momentum of air plus droplets is conserved. *Cotton et al.* [2011, see their Figure 2.2] endorsed this interpretation.

However, the reactive motion explanation appears counter to our knowledge of atmospheric processes: indeed, unlike a rocket which accelerates by reactive motion, i.e., by *internal* forces between the rocket and the expelled fuel, droplets upon condensation of water vapor are accelerated downward by a recognized and *external* force—gravity. (As another example, consider a block of ice melting on a table made of open mesh. The block does not accelerate upwards as the melt water streams down.)

Bannon [2002] mentioned the disagreement with *Ooyama* [2001] but did not identify either its cause or implications. If, as suggested by *Cotton et al.* [2011], the formulation of B02 obeys momentum conservation, does the contrasting formulation of O01, used in global atmospheric models [*Satoh*, 2014], violate it? While some authors have argued that the reactive motion term in equation (14) is usually small [*Monteiro and Torlaschi*, 2007], *Cotton et al.* [2011] concluded that this term warranted further study. Irrespective of its magnitude, resolving the discrepancy between the formulations of O01 and B02 is necessary for correct employment of the fundamental equations of momentum conservation to a moist atmosphere. We undertake such an analysis below considering O01 in section 2, B02 in section 3 and summarizing the conclusions in section 4.

2. Derivations of *Ooyama* [2001]

2.1. Equations of Motion and Momentum Equations

Unlike the meteorological literature where any equation involving air acceleration can apparently be called a "momentum equation," the physics literature is more selective. For example, in the physical textbook *Fluid Mechanics* of *Landau and Lifshitz* [1987] there is no mention of momentum equations. The equations formulated by Euler and Navier and Stokes are *equations of motion*. Below the momentum equations denote only those equations that describe change of momentum ($\rho_a \mathbf{v}_a$ or $\rho_c \mathbf{v}_c$): for example, equations (9) and (10) in Figure 1. Equations that describe any change of velocity ($\mathbf{v}_a$ or $\mathbf{v}_c$) are referred to as "equation of motion": for example, equations (5)–(8) in Figure 1.

Both O01 and B02 begin their derivations from a *momentum equation* for the system *moist air plus droplets*, see equations (9) and (10) in Figure 1. Since none of these previous authors justify their basic equations, we begin by showing how they could be derived. The equations of motion for moist air and condensate in their general form can be written as follows:

$$\rho_a \frac{D_a \mathbf{v}_a}{Dt} - \mathbf{F}_a = 0, \tag{16}$$

$$\rho_c \frac{D_c \mathbf{v}_c}{Dt} - \mathbf{F}_c = 0. \tag{17}$$

Here $\mathbf{F}_a$ and $\mathbf{F}_c$ are the volume-specific forces acting on gaseous air and condensate, respectively, and the material derivatives are defined by equation (4) in Figure 1.

The use of equation (17) for condensate particles by both O01 and B02 implies an assumption essential for our subsequent analysis, namely, that all condensate particles in a local volume have the same velocity $\mathbf{v}_c$. Indeed, applying Newton's second law to N droplets contained in an atmospheric volume $\tilde{V}_c$ yields

$$\frac{1}{\tilde{V}_c} \sum_{i=1}^{N} m_i \frac{d\mathbf{v}_{ci}}{dt} = \frac{1}{\tilde{V}_c} \sum_{i=1}^{N} \mathbf{f}_{ci}, \quad \rho_c \equiv \frac{1}{\tilde{V}_c} \sum_{i=1}^{N} m_i, \quad \mathbf{F}_c \equiv \frac{1}{\tilde{V}_c} \sum_{i=1}^{N} \mathbf{f}_{ci}, \tag{18}$$

where m_i, $\mathbf{v}_{ci}$, and $\mathbf{f}_{ci}$ are the mass, velocity, and force acting on the ith droplet, respectively. Putting $d\mathbf{v}_{ci}/dt \equiv (\mathbf{v}_{ci} \cdot \nabla)\mathbf{v}_{ci}$ (see equation (4) in Figure 1) we find that equation (18) is equivalent to equation (17) if and only if $\mathbf{v}_{ci} = \mathbf{v}_c$: i.e., all droplets have the same velocity. If there are *discrete* types of condensate particles (ice, snow, and rain) of different size, for each such type a separate equation similar to equation (17) is used. As we will see below, the main theoretical problem is presented by particles that have a *continuous* velocity distribution changing their velocity rapidly in a local volume.

Generally, the nonlinear equations (16) and (17) presume the existence of atmospheric volumes $\tilde{V}_a$ and $\tilde{V}_c$ within which the velocities of, respectively, air and condensate vary insignificantly—such that all the air within $\tilde{V}_a$ and all droplets within $\tilde{V}_c$ can be assumed to possess the same velocities—$\mathbf{v}_a$ for air and $\mathbf{v}_c$ for condensate.

The continuity equations (1)–(3), see Figure 1, in O01 and B02 are the same. Let us multiply equations (1) and (2) by, respectively, $\mathbf{v}_a$ and $\mathbf{v}_c$ and sum them up with the respective equations of motion (16) and (17). After rearranging the terms, we obtain the momentum equations for gaseous air and condensate:

$$\frac{D_a \rho_a \mathbf{v}_a}{Dt} + \rho_a \mathbf{v}_a \left(\nabla \cdot \mathbf{v}_a\right) - \mathbf{F}_a - \mathbf{v}_a \dot{\rho}_a = 0, \tag{19}$$

$$\frac{D_c \rho_c \mathbf{v}_c}{Dt} + \rho_c \mathbf{v}_c \left(\nabla \cdot \mathbf{v}_c\right) - \mathbf{F}_c - \mathbf{v}_c \dot{\rho}_c = 0. \tag{20}$$

The first two terms in equations (19) and (20) represent, respectively, change of momentum, per unit volume, of a certain amount of air and condensate. Indeed, consider an air parcel with mass $m_a = \rho_a \tilde{V}_a$ occupying volume $\tilde{V}_a$. Its momentum is $m_a \mathbf{v}_a$. Change of momentum, taken per unit air volume, is equal to

$$\frac{1}{\tilde{V}_a} \frac{d\left(m_a \mathbf{v}_a\right)}{dt} = \frac{d\rho_a \mathbf{v}_a}{dt} + \frac{\rho_a \mathbf{v}_a}{\tilde{V}_a} \frac{d\tilde{V}_a}{dt}. \tag{21}$$

Summing equations (19) and (20) using equation (3) and recalling that $\mathbf{W} \equiv \mathbf{v}_c - \mathbf{v}_a$, we obtain an equation for the change of the total momentum, per unit volume, of the system with a constant total mass: gas occupying volume $\tilde{V}_a$ and condensate particles occupying volume $\tilde{V}_c$ that coincide at the considered moment of time, $\tilde{V}_a = \tilde{V}_c$:

$$\frac{D_a \rho_a \mathbf{v}_a}{Dt} + \rho_a \mathbf{v}_a \left(\nabla \cdot \mathbf{v}_a\right) + \frac{D_c \rho_c \mathbf{v}_c}{Dt} + \rho_c \mathbf{v}_c \left(\nabla \cdot \mathbf{v}_c\right) = \mathbf{F}_a + \mathbf{F}_c - \mathbf{W}\dot{\rho}_a. \quad (22)$$

The constancy of mass for this system is dictated by the equality $\dot{\rho}_a(\mathbf{r}, t) = -\dot{\rho}_c(\mathbf{r}, t)$; see equation (3) in Figure 1. This equation prescribes that gas with velocity $\mathbf{v}_a$ turns into a condensate particle with velocity $\mathbf{v}_c$ locally (i.e., at the same coordinate $\mathbf{r}$) and instantaneously (i.e., at the same time point t). Thus, total mass within the considered volumes $\tilde{V}_a$ and $\tilde{V}_c$ is conserved.

The difference in formulations of B02 and O01 pertain to their logic in specifying forces $\mathbf{F}_a$ and $\mathbf{F}_c$. Both authors agree that the external forces to consider are gravity (which acts on both air and condensate) and the macroscopic pressure gradient (which acts on the air alone but can be neglected for droplets because of their small size).

Ooyama [2001] further assumed that whatever forces exist between air and condensate, they are of equal magnitude but opposite sign by Newton's third law and thus should cancel and vanish in the sum of $\mathbf{F}_a + \mathbf{F}_c$ in the right-hand part of equation (22); see equations (5) and (6) in Figure 1:

$$\mathbf{F}_a = -\nabla p + \rho_a \mathbf{g} + \mathbf{F}_{ac}, \quad \mathbf{F}_c = \rho_c \mathbf{g} + \mathbf{F}_{ca}, \quad \mathbf{F}_{ac} = -\mathbf{F}_{ca}. \quad (23)$$

Here $\mathbf{F}_{ac}$ and $\mathbf{F}_{ca}$ are the forces exerted on the air by the condensate and on the condensate by the air, respectively; $\mathbf{F}_{ac} = -\mathbf{F}_{ca}$ according to Newton's third law. Using equation (23), summing up equations (16) and (17) (and using the assumption for condensate acceleration (11) to be discussed below), we obtain Ooyama's equation (13); see green dashed arrows in Figure 1. This equation of motion lacks a reactive motion term.

On the other hand, from equation (22) we see that even if forces are not specified in the equations of motions (16) and (17), the reactive motion term appears in the total momentum equation with the minus sign (cf. also equations (7) and (9) in Figure 1). What is the meaning of this term?

The momentum equation (22), based on the equations of motion (16) and (17), describes a system composed of two types of objects: air with velocity $\mathbf{v}_a$ and droplets with velocity $\mathbf{v}_c$. Since the continuity equation demands that $\dot{\rho}_a(\mathbf{r}, t) = -\dot{\rho}_c(\mathbf{r}, t)$, equation (3) in Figure 1, these objects must change their velocity from $\mathbf{v}_a$ to $\mathbf{v}_c$ or vice versa. However, in reality, such changes cannot be instantaneous. Accordingly, when phase transitions are occurring in the atmosphere, there exist objects with intermediate velocities (between $\mathbf{v}_a$ and $\mathbf{v}_c$). As water vapor condenses into a droplet, the latter has an initial velocity equal to that of local air, Figure 2a. The droplet then is accelerated under the action of gravity until it reaches terminal velocity $\mathbf{W}$, when gravity is compensated by the reaction force of the air. For $W = 5$ m s^{-1} this acceleration takes about $t_T \sim W/g$ half a second and ceases at a distance of about $l_T \sim W^2/g \sim 3$ m below the point of condensation. If l_T is much smaller than the vertical size of the considered atmospheric volume $\tilde{V}_a$, most droplets born within this volume will still reside in that volume as they reach terminal velocity. As noted by *Ooyama* [2001], as soon as they reach terminal velocity, such droplets are *reclassified* as droplets having velocity $\mathbf{v}_c$, from which point on they obey the equation of motion (17); see also equation (6) in Figure 1. O01 implicitly assumes that droplets of intermediate velocity do not interact with either other droplets or air. Only droplets with velocity $\mathbf{v}_c$ exert force $\mathbf{F}_{ac}$ on the air and experience force $\mathbf{F}_{ca}$ from the air.

Thus, in the formulation of *Ooyama* [2001], $-\mathbf{W}\dot{\rho}_a$ approximates the external force of gravity that is accelerating the newly formed droplets. Formally, it is as if gas with velocity $\mathbf{v}_a$ disappears from the system and then instantaneously reappears as a droplet with velocity $\mathbf{v}_c$ having in the meantime been accelerated by an external force—gravity. Therefore, the corresponding term $-\mathbf{W}\dot{\rho}_a$ describing this acceleration finds itself in equation (9) among other external forces acting on the system *gas plus droplets with velocity* $\mathbf{v}_c$. However, since $-\mathbf{W}\dot{\rho}_a$ *does not* accelerate either air or droplets with velocity $\mathbf{v}_c$ (it only accelerates droplets with intermediate velocity), it is absent from the equations of motion for air and droplets with velocity $\mathbf{v}_c$, i.e., from equations (5) and (6) in Figure 1.

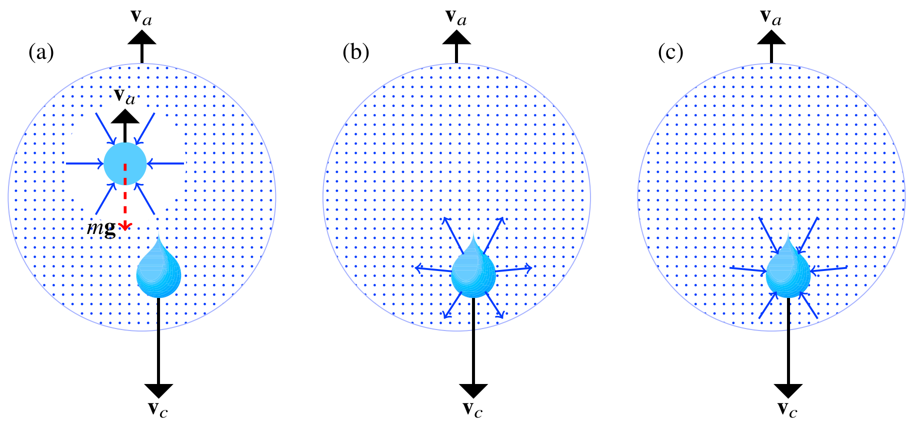

Figure 2. Different assumptions about interaction between air and condensate particles in the formulations of (a) O01 and (b, c) B02. The big circle represents an air volume $\tilde{V}_a$ moving with velocity $\mathbf{v}_a$ and filled with water vapor (dots). Thin arrows directed to or from the droplets represent condensation (Figures 2a and 2c) or evaporation (Figure 2b), respectively. In Figure 2a, the droplet forms anew from water vapor with velocity $\mathbf{v}_a$ and is accelerated by gravity (red dashed arrow).

2.2. Acceleration of Condensate Particles

While gravity is responsible for keeping the vertical droplet velocity distinct from the vertical velocity of air, no such forces exist in the horizontal plane. Thus, O01 postulated that horizontal velocities $\mathbf{u}_a$ and $\mathbf{u}_c$ of air and droplets coincide, while vertical velocities $\mathbf{w}_a$ and $\mathbf{w}_c$ differ by $\mathbf{W}$:

$$\mathbf{u}_a = \mathbf{u}_c, \quad \mathbf{W} \equiv \mathbf{v}_c - \mathbf{v}_a = \mathbf{w}_c - \mathbf{w}_a, \tag{24}$$

where $\mathbf{v}_c = \mathbf{u}_c + \mathbf{w}_c$ and $\mathbf{v}_a = \mathbf{u}_a + \mathbf{w}_a$. Furthermore, *Ooyama* [2001, see equation (3.10)] assumed that $\mathbf{W}$ does not vary along the droplet path:

$$\frac{D_c \mathbf{W}}{Dt} = 0. \tag{25}$$

These two assumptions combined yield equation (11) for droplet acceleration; see Figure 1. In the equation of motion (13) the condensate acceleration plays the role of a drag force imposed by the droplets on the air.

These assumptions are justified when the interaction $\mathbf{F}_{ca}$ between the air and condensate equalizes velocities more rapidly than they are changed by macroscopic gradients. Consider the case when the interaction between air and droplets is given by the Stokes force $\mathbf{f}_S$. The Stokes force is proportional to the velocity difference $\Delta\mathbf{v}$ between air and droplets:

$$f_S = 3\pi\rho_a \nu d \Delta v, \quad a_S \equiv \frac{f_S}{m}, \quad \tau_S \equiv \frac{\Delta v}{a_S} = \frac{1}{18}\frac{\rho_l}{\rho_a}\frac{d^2}{\nu}, \tag{26}$$

where d is droplet diameter, ν is the kinematic viscosity of air, a_S is acceleration of a spherical droplet with mass $m = \pi\rho_l d^3/6$, $\rho_l = 10^3$ kg m^{-3} is the density of liquid water, τ_S is the time scale at which the Stokes force equalizes the velocities of air and condensate.

Let the horizontal air velocity u_a be changed by Δu_a over a typical vertical scale $h \gtrsim 10^2$ m. Then condition

$$\tau_S \ll \frac{h}{W_S}, \quad \tau_S \ll \sqrt{\frac{h}{g}}, \quad W_S \equiv \tau_S g, \tag{27}$$

ensures that for a droplet falling with terminal velocity W_S we have $|u_c - u_a| \ll |\Delta u_a|$. For small droplets with $d < 0.1$ mm obeying the Stokes law (26) we have $\tau_S \sim 0.05$ s assuming $\nu \sim 10^{-5}$ m^2 s^{-1} and $\rho_l/\rho_a \sim 10^3$. The terminal velocity W and the corresponding time scale $\tau \equiv W/g$ of the largest drops with $d \sim 1$ cm are a factor of 10^3 smaller than, respectively, W_S (27) and τ_S (26)—due to turbulence effects [*Gunn and Kinzer*, 1949]. Thus, condition (27) is fulfilled even for these largest drops when τ_S and W_S are replaced by, respectively, $\tau \sim 10^{-3}\tau_S$ and $W \sim 10^{-3}W_S$.

These estimates show that the interaction between air and droplets rapidly equalizes their horizontal velocities justifying the assumption $\mathbf{u}_a = \mathbf{u}_c$ of O01. In the vertical plane, due to the presence of gravity, it aligns droplet velocity with the terminal velocity defined from the balance of the Stokes force and gravity neglecting the droplet acceleration. This justifies the assumption made by O01 that $\mathbf{W}$ does not change along the droplet path and can be parameterized.

3. Derivations of *Bannon* [2002]

3.1. Equations of Motion and Momentum Equations

In the logic of B02, the terms to appear in the right-hand part of the total momentum equation should only be the external forces (pressure gradient $-\nabla p$ acting on air and gravity $(\rho_a + \rho_c)\mathbf{g}$ acting on air and droplets), see equation (10) in Figure 1; they cannot include anything dependent on the interaction between the droplets and the air since that would be an internal force.

Thus, B02 starts from equation (10), which represents equation (22) without $-\mathbf{W}\dot{\rho}_a$ (Figure 1). Subtracting from equation (10), the continuity equations (1) and (2) multiplied, respectively, by $\mathbf{v}_a$ and $\mathbf{v}_c$, and using equation (12), which assumes that droplets do not accelerate, B02 obtains his resulting equation of motion (14) for moist air; see black solid arrows in Figure 1, right column. This equation contains the reactive motion term, which B02 interpreted as the upward acceleration the air acquires when droplets with velocity $\mathbf{v}_c$ begin to fall.

However, this derivation appears to conflict with Newton's third law (Figure 1). Indeed, there are only two ways to remove $-\mathbf{W}\dot{\rho}_a$ from the right-hand side of the total momentum equation: it is to add the same term, but with an opposite sign, to the formulation of either $\mathbf{F}_a$ or $\mathbf{F}_c$ in the equations of motion (16) or (17) (i.e., of air or condensate). B02 chooses the former without any stated justification, cf. equation (23) of O01:

$$\mathbf{F}_a = -\nabla p + \rho_a \mathbf{g} + \mathbf{F}_{ac} + \mathbf{W}\dot{\rho}_a, \quad \mathbf{F}_c = \rho_c \mathbf{g} + \mathbf{F}_{ca}. \tag{28}$$

Note that if one separately adds $-\mathbf{v}_a\dot{\rho}_a$ to $\mathbf{F}_a$ and $\mathbf{v}_c\dot{\rho}_a$ to $\mathbf{F}_c$, the resulting equations of motion (16) and (17) will be flawed. Such equations, similar to the equation $\mathrm{d}w/\mathrm{d}t = -(w/m)\mathrm{d}m/\mathrm{d}t$ on p. 1972 of B02 aimed to clarify the reactive motion during condensation, violate Galilean invariance by producing different acceleration in different inertial frames of reference. Such errors are not infrequent in considerations of systems with variable mass; for a discussion see, e.g., *Plastino and Muzzio* [1992] and *Irschik and Holl* [2004].

The question is, if $\mathbf{W}\dot{\rho}_a$ is an internal force acting between the air and condensate, which is the reason it did not show up in the right-hand part of the total momentum equation (10), then why does not this force obey Newton's third law? In other words, why is not this force present with opposite signs in the equations of motion for both condensate and droplets?

This apparent contradiction can be resolved and the true physical meaning of the reactive motion term clarified if we once again take into consideration the existence of objects with intermediate velocity. In the case of evaporation, Figure 2b, such an object is the water vapor that has just evaporated from a droplet with velocity $\mathbf{v}_c$. This water vapor initially has velocity $\mathbf{v}_c$ equal to that of the droplet it evaporated from. It then interacts with local air in the volume $\tilde{V}_a$. The result of this interaction is that their velocities equalize. This process is equivalent to an inelastic collision between an amount of air (water vapor) with velocity $\mathbf{v}_c$ and another amount of air with velocity $\mathbf{v}_a$, which subsequently move with the same velocity. Since during evaporation $\dot{\rho}_a > 0$, and the droplets have a downward velocity relative to the air, the reactive motion term in equations (7) and (14) describes the downward acceleration of air as it mixes with water vapor.

For acceleration $\mathbf{a}_v$ of just evaporated water vapor of mass m_v we can write

$$\frac{m_v \mathbf{a}_v}{\tilde{V}_a} = \mathbf{F}_{va}, \quad \mathbf{F}_{va} = -\mathbf{F}_{av}, \quad \mathbf{F}_{av} = \mathbf{W}\dot{\rho}_a. \tag{29}$$

Here $\mathbf{F}_{va}$ is the force, per unit volume, exerted by the air on the evaporating water vapor; $\mathbf{F}_{av}$ is, by Newton's third law, the opposite force exerted by the water vapor on the local air. Since acceleration of this water vapor occurs instantaneously, as dictated by the continunity equation (3), the steady state mass m_v of such water vapor with intermediate velocity approaches zero; their product is finite and equals the reactive motion term, equation (29). Thus, $\mathbf{W}\dot{\rho}_a$ in equations (7) and (14) does not violate Newton's third law as it describes interaction of local air not with the droplets of velocity $\mathbf{v}_c$ but with the "just evaporated" water vapor of intermediate velocity.

Notably, evaporation does not affect droplet motion, since evaporating water vapor has the same velocity as the droplet from which it evaporates. Thus, during evaporation, the equation of motion for droplets (8) does not include $\mathbf{W}\dot{\rho}_a$.

One could formally consider condensation to occur not by the birth of new droplets but by the same "inelastic collision" of water vapor molecules with preexisting droplets having velocity $\mathbf{v}_c$, Figure 2c. In this case it is the droplet velocity that is affected by interaction with the vapor of different velocity: the droplets will acquire an upward acceleration. The reactive motion term will then be present in the equation of motion for the droplets (8) but absent from the equation of motion for air (7). The resulting equation of motion for moist air (14) which contains their sum would remain the same. However, such a representation rules out the birth of new droplets leaving the origins and numbers of the preexisting droplets unexplained. In addition, rate of condensation can be proportional to the vertical velocity of the local air, while the number of droplets depends on their size. Therefore, requiring condensation to occur on preexisting droplets is equivalent to postulating a link between two unrelated parameters. This means that the reactive motion term cannot be justified for condensation.

3.2. Acceleration of Condensate

In the presence of condensation by inelastic collision, Figure 2c, the reactive motion term would reside in the equation of motion for droplets. Without taking into account other forces, the reactive force would accelerate droplets upward. Thus, neglecting droplet acceleration, see equation (12), while keeping a reactive motion term, appears contradictory and unjustified. With a typical rainfall rate $P \sim 10^{-3}$ kg m^{-2}s^{-1} (~4 mm h^{-1}) the mean condensation rate in an atmospheric column of height $h \sim 10^3$ m is $\dot{\rho}_a \sim -P/h \sim -10^{-6}$ kg m^{-3}s^{-1}. Meanwhile with $\rho_c \sim 10^{-4}$ kg m^{-3} and $|\partial v_a/\partial z| \sim \Delta v_a/h$ with $\Delta v_a \sim 10$ m s^{-1} we have $\rho_c|\partial v_a/\partial z| \sim |\dot{\rho}_a|$. This implies that the reactive motion term $\mathbf{W}\dot{\rho}_a$ and condensate acceleration $\rho_c(\mathbf{W}\cdot\nabla)\mathbf{v}_a$, equation (11), can be of similar magnitude.

Finally, if condensation occurred only on preexisting droplets, the droplets would grow in size as they fell. Since terminal velocity depends on droplet size, the assumption of zero acceleration of droplets adopted in B02, see equation (12), would be invalid.

4. Conclusions

While a valid equation of motion must conform to Galilean invariance and to fundamental conservation laws, it cannot be derived from those constraints alone. We have seen that the equation of total momentum (22) represents a sum of two equations of motions and two continuity equations each multiplied by the respective velocity of air or condensate. Accordingly, the momentum equation carries no additional information about system dynamics than what is already contained in the equations of motion and continuity (the latter only providing an approximation since they assume instantaneous velocity changes between air and droplets at phase transitions). Employing the total momentum equation without explicit consideration of the equations of motion (16) and (17) resulted in confusion concerning the reactive motion term of B02.

For droplets born during condensation the reactive motion term describes their downward acceleration, Figure 2a. For evaporation from droplets the term describes the downward acceleration of air, Figure 2b. For condensation on preexisting droplets the term describes the upward acceleration of droplets, Figure 2c. In none of these cases, contrary to previous suggestions, does the reactive motion term describe the upward acceleration of air. Such a process does not exist.

A generally applicable equation taking these processes into account is (cf. equations (13) and (14))

$$(\rho_a + \rho_c)\frac{D_a\mathbf{v}_a}{Dt} = -\nabla p + \left(\rho_a + \rho_c\right)\mathbf{g} - \rho_c(\mathbf{W}\cdot\nabla)\mathbf{v}_a + \mathbf{W}\dot{\rho}_a^{+}. \quad (30)$$

Here $\dot{\rho}_a^{+} > 0$ is the rate of evaporation (e.g., from big drops under the cloud base). This term will be absent where condensation occurs ($\dot{\rho}_a < 0$). Equation (30) accounts for the condensate drag neglected by B02. As we estimated in section 2.2, this drag formulated by O01 correctly represents the interaction between air and condensate under most atmospheric conditions.

Phase transitions incur abrupt changes of fluid properties that occur near instantaneously compared to the typical time scales of air motion. Here we have considered how the instantaneous formation of condensate affects the motion of moist air. However, the dynamic effect of phase transitions is not confined to the interaction between air and condensate. An additional effect is the local pressure perturbation that arises during condensation; see Figure 2a. In a hydrostatic atmosphere such perturbations can cause larger scale pressure adjustments. These processes can transform potential energy contained in local condensation-induced pressure perturbations into the potential energy of pressure gradients able to drive macroscopic air motions. Clarifying this effect of condensation is, in our view, a promising research perspective.

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