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Eccentric Point Particle Binaries In The Linear Regime Of The Characteristic Formulation Of General Relativity

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Contents

1. Objectives

- Examine in full details the boundary conditions on the world tubes generated by the orbits of the binary system.
- Show that in the linearized regime of the Einstein field equations in the Characteristic Formulation, the power emitted by a binary system computed via News function is in agreement with the Peters and Mathews expression obtained in the 1960s.

2. Linearized Regime

- [Bondi et al., 1962, Sachs, 1962], Bondi-Sachs metric

$$
\begin{aligned}
ds^2 = &-\left(\left(1+\frac{w}{r}\right)e^{2\beta} - r^2 h_{AB}U^A U^B\right) du^2 - 2e^{2\beta} du dr \\
&- 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B,
\end{aligned} \tag{1}
$$

- At the linearized regime, perturbing the Minkowski metric, (1) becomes

$$
g_{\mu\nu} =: \begin{pmatrix}
1-\frac{w}{r}-2\beta & -1-2\beta & -\frac{r^2(U+\overline{U})}{1+\zeta\overline{\zeta}} & i\frac{r^2(U-\overline{U})}{1+\zeta\overline{\zeta}} \\
-1-2\beta & 0 & 0 & 0 \\
-\frac{r^2(U+\overline{U})}{1+\zeta\overline{\zeta}} & 0 & \frac{2r^2(2+J+\overline{J})}{(1+\zeta\overline{\zeta})^2} & -\frac{2ir^2(J-\overline{J})}{(1+\zeta\overline{\zeta})^2} \\
i\frac{r^2(U-\overline{U})}{1+\zeta\overline{\zeta}} & 0 & -\frac{2ir^2(J-\overline{J})}{(1+\zeta\overline{\zeta})^2} & -\frac{2r^2(-2+J+\overline{J})}{(1+\zeta\overline{\zeta})^2}
\end{pmatrix}.
$$

- where

$$
J = \frac{h_{AB}q^A q^B}{2}, \quad \overline{J} = \frac{h_{AB}\overline{q}^A\overline{q}^B}{2}, K = \frac{h_{AB}\overline{q}^A q^B}{2}, \quad U = U^A q_A, \quad \overline{U} = U^A \overline{q}_A, \tag{2}
$$

are spin-weighted scalars [Winicour, 2012, Bishop et al., 1997, Bishop et al., 2005]

- q^A are tangent vectors to the unit sphere, in stereographic coordinates they are

$$q^A = \frac{(1+\zeta\bar{\zeta})}{2}\left(\delta^A{}_3 + i\delta^A{}_4\right), \quad q^\mu = (0,0,q^A). \tag{3}$$

[Gómez et al., 1997, Newman and Penrose, 1966, Goldberg et al., 1967]

- where

$$\zeta_N = \tan\left(\frac{\theta}{2}\right)e^{i\phi}, \ \ \theta \in [0,\pi/2], \ \ \zeta_S = \cot\left(\frac{\theta}{2}\right)e^{i\phi}, \ \ \theta \in [0,\pi/2], \\ \phi \in [0,2\pi) \tag{4}$$

- The field equations are decomposed as hypersurface, evolution and constrains [Bishop et al., 1997, Bishop, 2005, Reisswig et al., 2010]

$$E_{22} = 0, \ \ E_{2A}q^A = 0, \quad E_{AB}h^{AB} = 0, \tag{5a}$$

$$E_{AB}q^Aq^B = 0, \tag{5b}$$

$$E_{11} = 0, \ \ E_{12} = 0, \quad E_{1A}q^A = 0, \tag{5c}$$

- where

$$E_{\mu\nu} = R_{\mu\nu} - 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \quad \text{with} \quad E_{\mu\nu} = 0 \tag{6}$$

- The field equations read [Bishop, 2005]

$$8\pi T_{22} = \frac{4\beta_{,r}}{r}, \tag{7a}$$

$$8\pi T_{2A}q^A = \frac{\bar{\eth} J_{,r}}{2} - \eth\beta_{,r} + \frac{2\eth\beta}{r} + \frac{\left(r^4 U_{,r}\right)_{,r}}{2r^2}, \tag{7b}$$

$$\begin{aligned} 8\pi\left(h^{AB}T_{\mathrm{AB}} - r^2 T\right) &= -2\eth\bar{\eth}\beta + \frac{\eth^2\bar{J} + \bar{\eth}^2 J}{2} + 4\beta - 2w_{,r} \\ &\quad + \frac{\left(r^4\left(\bar{\eth}U + \eth\bar{U}\right)\right)_{,r}}{2r^2}, \end{aligned} \tag{7c}$$

$$8\pi T_{AB}q^A q^B = -2\eth^2\beta + \left(r^2\eth U\right)_{,r} - \left(r^2 J_{,r}\right)_{,r} + 2r\left(rJ\right)_{,ur}, \tag{7d}$$

$$8\pi\left(\frac{T}{2} + T_{11}\right) = \frac{\eth\bar{\eth}w}{2r^3} + \frac{\eth\bar{\eth}\beta}{r^2} - \frac{\left(\eth\bar{U} + \bar{\eth}U\right)_{,u}}{2} + \frac{w_{,u}}{r^2} + \frac{w_{,rr}}{2r} - \frac{2\beta_{,u}}{r}, \tag{7e}$$

$$8\pi\left(\frac{T}{2} + T_{12}\right) = \frac{\eth\bar{\eth}\beta}{r^2} - \frac{\left(r^2\left(\eth\bar{U} + \bar{\eth}U\right)\right)_{,r}}{4r^2} + \frac{w_{,rr}}{2r}, \tag{7f}$$

$$\begin{aligned} 8\pi T_{1A}q^A &= \frac{\bar{\eth}J_{,u}}{2} - \frac{\eth^2\bar{U}}{4} + \frac{\eth\bar{\eth}U}{4} + \frac{1}{2}\left(\frac{\eth w}{r}\right)_{,r} - \eth\beta_{,u} + \frac{\left(r^4 U\right)_{,r}}{2r^2} \\ &\quad - \frac{r^2 U_{,ur}}{2} + U. \end{aligned} \tag{7g}$$

- Where the differential operators ð and $\bar{ð}$ are the projections of the covariant derivatives onto q^A vectors and satisfy

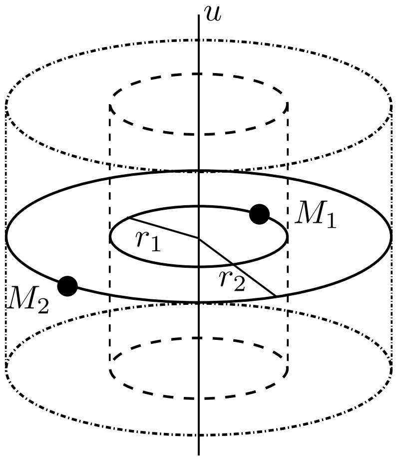
$$ð \; {}_s\Psi = (1+\zeta\bar{\zeta}) \; {}_s\Psi_{,\bar{\zeta}} + s\zeta \; {}_s\Psi, \tag{8a}$$

$$\bar{ð} \; {}_s\Psi = (1+\zeta\bar{\zeta}) \; {}_s\Psi_{,\zeta} - s\bar{\zeta} \; {}_s\Psi, \tag{8b}$$

$$[ð, \bar{ð}] \; {}_s\Psi = -2s \; {}_s\Psi. \tag{8c}$$

3. Solution for a eccentric binary system

- .



- $$
{}_sf = \sum_{l,m} \Re\Big(f_{lm} e^{i|m|\tilde{\phi}} \Big) \, \eth^s Z_{lm}
$$

  where ${}_sf = \{\beta, w, J, \overline{J}, U, \overline{U}\}$,
  $\tilde{\phi} := \tilde{\phi}(u)$

- Two (for equal masses) or three (different masses) empty regions

- Two ways to solve this boundary problem:
  - Vacuum solutions + jumps in $g_{\mu\nu}$ and in $g_{\mu\nu,\gamma} \Rightarrow T_{\mu\nu}$
  - Vacuum solutions + $T_{\mu\nu} \Rightarrow$ jumps in $g_{\mu\nu}$ and in $g_{\mu\nu,\gamma}$.

- The solutions for the radial part of the equation (7) for the vacuum, for $l=2$ and $m \neq 0$, are

$$
\begin{aligned}
\beta_{2m}(r) &= D_{1\beta 2m}, \\
J_{2m}(r) &= \frac{2iD_{1\beta 2m}}{\dot{\tilde{\phi}} r |m|} - \frac{D_{1J2m}(\dot{\tilde{\phi}} r |m| - 1)(\dot{\tilde{\phi}} r |m| + 1)}{6r^3} \\
&\quad - \frac{iD_{2J2m} e^{2i\dot{\tilde{\phi}} r |m|}(\dot{\tilde{\phi}} r |m| + i)^2}{8\dot{\tilde{\phi}}^5 r^3 |m|^5} + \frac{D_{3J2m}(\dot{\tilde{\phi}} r |m| - 3i)}{\dot{\tilde{\phi}} r |m|}, \\
U_{2m}(r) &= \frac{2D_{1\beta 2m}(\dot{\tilde{\phi}} r |m| + 2i)}{\dot{\tilde{\phi}} r^2 |m|} - \frac{iD_{3J2m}\left(\dot{\tilde{\phi}}^2 r^2 |m|^2 + 6\right)}{\dot{\tilde{\phi}} r^2 |m|} \\
&\quad - \frac{D_{1J2m}\left(2\dot{\tilde{\phi}}^2 r^2 |m|^2 + 4i\dot{\tilde{\phi}} r |m| + 3\right)}{6r^4} - \frac{D_{2J2m} e^{2i\dot{\tilde{\phi}} r |m|}(2\dot{\tilde{\phi}} r |m| + 3i)}{8\dot{\tilde{\phi}}^5 r^4 |m|^5}, \\
w_{2m}(r) &= -10rD_{1\beta 2m} + 6rD_{3J2m}(2 + i\dot{\tilde{\phi}} r |m|) \\
&\quad - \frac{iD_{1J2m}((1+i)\dot{\tilde{\phi}} r |m| - i)(1 + (1+i)\dot{\tilde{\phi}} r |m|)}{r^2} - \frac{3iD_{2J2m} e^{2i\dot{\tilde{\phi}} r |m|}}{4\dot{\tilde{\phi}}^5 r^2 |m|^5},
\end{aligned}
$$

- $D_{nflm} := D_{nflm}(u)$ in order to allow ellipticity in the orbits.

- Boundary conditions in the world tubes

$$\begin{aligned}&[g_{11}]_{r_i}=0,\ \ [g_{12}]_{r_i}=\Delta g_{12}|_{r_i},\ \ [g_{1A}]_{r_i}=0,\ \ [g_{22}]_{r_i}=0,\\&[g_{2A}]_{r_i}=0,\ \ [g_{3\mu}]_{r_i}=0,\ \ [g_{4\mu}]_{r_i}=0,\end{aligned} \tag{10}$$

- In their first derivatives,

$$\left[g'_{\mu\nu}\right]_{r_i}=\Delta g'_{\mu\nu},\ \ \mu,\nu=1,\cdots 4, \tag{11}$$

where the brackets mean $[f(r)]_{r_i}=f(r)|_{r_i+\epsilon}-f(r)|_{r_i-\epsilon}$.

- Then, the jump in the metric coefficients are

$$\begin{aligned}[w_{lm}(r_j)]&=\Delta w_{jlm}, & [\beta_{lm}(r_j)]&=\Delta\beta_{jlm},\\ [J_{lm}(r_j)]&=0, & [U_{lm}(r_j)]&=0,\end{aligned} \tag{12}$$

- and for their first derivatives

$$\begin{aligned}[w'_{lm}(r_j)]&=\Delta w'_{jlm}, & [\beta'_{lm}(r_j)]&=\Delta\beta'_{jlm},\\ [J'_{lm}(r_j)]&=\Delta J'_{jlm}, & [U'_{lm}(r_j)]&=\Delta U'_{jlm},\end{aligned} \tag{13}$$

where $j=1,2$, and Δw_{jlm}, $\Delta\beta_{jlm}$, $\Delta w'_{jlm}$, $\Delta\beta'_{jlm}$, $\Delta J'_{jlm}$ and $\Delta U'_{jlm}$ are functions to be determined.

- Space-time flat at the vertex of the cones
- Regularity of the space-time at the null infinity.
- The solutions for the algebraic system of equations are,

$$\Delta\beta_{jlm} = b_{jlm}, \quad \Delta w_{jlm} = -2r_j b_{jlm}, \tag{14}$$

where b_{jlm} are constants.

- The last fact implies that $\Delta\beta'_{jlm} = 0$.
- We obtain that the jumps for the first derivative of the J_{lm} and U_{lm} functions are given by

$$\Delta J'_{jlm} = \frac{8\dot{\tilde{\phi}}^2 r_j b_{jlm} \left|m\right|^2}{(l-1)l(l+1)(l+2)}, \tag{15}$$

$$\Delta U'_{jlm} = 2b_{ilm}\left(\frac{1}{r_i^2} - \frac{4i\dot{\tilde{\phi}}|m|}{l(l+1)r_i}\right). \tag{16}$$

- If the system is a point particle binary, then

$$b_{jlm} = 2\int_{I_j} dr \int_0^{2\pi} d\tilde{\phi} \int_\Omega d\Omega \; {}_0Z_{lm} \; \rho \; \gamma^2, \quad I_j = (r_j - \frac{\epsilon}{2}, r_j + \frac{\epsilon}{2}) \tag{17}$$

- $$\rho = \frac{\delta(\theta - \pi/2)}{r^2}(M_1\delta(r - r_1)\delta(\phi - \tilde{\phi}) + M_2\delta(r - r_2)\delta(\phi - \tilde{\phi} - \pi)), \quad (18)$$

- [Peters and Mathews, 1963]

$$r_j = \mu_j d, \qquad \mu_j = \frac{\mu}{M_j}, \qquad \mu = \frac{M_1 M_2}{M_1 + M_2}, \quad (19)$$

$$\dot{\tilde{\phi}} = \frac{[(M_1 + M_2)a(1 - \epsilon^2)]^{1/2}}{d^2}, \quad d = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos\tilde{\phi}} \quad (20)$$

4. Gravitational Radiation

- The Bondi's News function was computed by [Bishop, 2005] and reads

$$\mathcal{N} = \lim_{r\to\infty}\left(-\frac{r^2 J_{,ur}}{2} + \frac{\eth^2\omega}{2} + \eth^2\beta\right). \tag{21}$$

- Thus

$$\begin{aligned}\mathcal{N} = \frac{1}{2}\sqrt{\frac{3}{2}}\Big[\,{}_2Z_2\ {}_{-2}\Re\left(e^{2i\tilde{\phi}}\left(D_{1\mathrm{J2\text{-}2+}}\right)'(u)\right) + \ {}_2Z_2\ {}_2\Re\left(e^{2i\tilde{\phi}}\left(D_{1\mathrm{J22+}}\right)'(u)\right)\Big]\\ + 2i\sqrt{\frac{2}{3}}\Big[\,{}_2Z_2\ {}_{-2}\Re\left(e^{2i\tilde{\phi}}\dot{\tilde{\phi}}^3 D_{2\mathrm{J2\text{-}2+}}(u)\right) + \ {}_2Z_2\ {}_2\Re\left(e^{2i\tilde{\phi}}\dot{\tilde{\phi}}^3 D_{2\mathrm{J22+}}(u)\right)\Big]\end{aligned} \tag{22}$$

- The power emitted by the system is computed as

$$P = \frac{dE}{du} = \int_\Omega d\Omega\,|\mathcal{N}|^2 \tag{23}$$

- For equal masses and in the very low velocity limit, $\gamma \simeq 1$, one obtains

$$P = -\frac{64\,(M_0)^5\left(\epsilon\cos\tilde{\phi}+1\right)^6}{5a^5\left(\epsilon^2-1\right)^5} - \frac{16\,(M_0)^5\epsilon^2\sin^2\tilde{\phi}\left(\epsilon\cos\tilde{\phi}+1\right)^4}{15a^5\left(\epsilon^2-1\right)^5} \tag{24}$$

that is just the Peters and Mathews expression for the Power emitted by a point particle binary.

- For circular orbits and different masses, the News reads

$$\mathcal{N} = 2i\sqrt{\frac{2}{3}}\left[\ {}_2Z_2\ {}_{-2}\Re\left(e^{2i\tilde{\phi}}\dot{\tilde{\phi}}^3 D_{2\text{J2-2+}}\right) +\ {}_2Z_2\ {}_2\Re\left(e^{2i\tilde{\phi}}\dot{\tilde{\phi}}^3 D_{2\text{J22+}}\right)\right] \quad (25)$$

- Then, the power emitted is

$$\begin{aligned}\frac{dE}{du} &= \frac{32}{5}\nu^6\left(\mathcal{M}_{21}+\mathcal{M}_{22}\right)^2+\frac{2734}{315}\nu^8\left(\mathcal{M}_{31}-\mathcal{M}_{32}\right)^2\\ &+ \frac{57376}{3969}\nu^{10}\left(\mathcal{M}_{41}+\mathcal{M}_{42}\right)^2+\frac{4010276}{155925}\nu^{12}\left(\mathcal{M}_{51}-\mathcal{M}_{52}\right)^2+\cdots\end{aligned} \quad (26)$$

where, the red color indicates the contribution to the power by terms of different value of l and

$$\mathcal{M}_{lj} = M_j r_j^l (v_j^2+1), \quad (27)$$

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