OPTIMIZATION OF LOW-THRUST TRANSFERS BETWEEN GIVEN ORBITS IN A PERTURBED GRAVITY FIELD

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Abstract: *Low-thrust power-limited transfers between given orbits in the presence of perturbations of different nature are considered. A simple method of obtaining the transfer trajectory is suggested; this method is based on the linearization of motion near reference orbits. A required calculation accuracy is reached by means of use of a proper number of the reference orbits. The method can be used in the case of a big number (up to several thousand) of revolutions around the attracting center without any averaging of motion. The following transfer types may be calculated using the method: from a given state to a given orbit with obtaining an optimal position of the entry into the orbit; from a given orbit to a given state with obtaining an optimal position of launch from the orbit; between two given orbits with obtaining optimal launch and entry positions. The suggested method also is applicable in the cases of partly given final orbit and of given constraints on the thrust direction. Due to the linearization the method does not give a globally optimal solution, the solution is locally optimal (i.e. optimal in each point of the transfer trajectory). This deficiency is compensated by the simplicity and wide applicability of the method. Numerical examples illustrating the method are given.*

Keywords: *Low Thrust, Spiral Trajectory, Transfer Optimization, Performance Index*

Nomenclature

$0, T$	=	initial and final instants
a	=	semi-major axis
$\mathbf{c} = \mathbf{r} \times \mathbf{v}$	=	integral of areas
$c = \lvert\mathbf{c}\rvert$		
e	=	eccentricity
$\mathbf{e} = \mathbf{v} \times \mathbf{c}/\mu - \mathbf{r}/r$	=	eccentricity vector
$\mathbf{f}_v = -\mu/r^3\,\mathbf{r}$	=	external forces
$\mathbf{f} = \mathbf{f}(\mathbf{x}) = \{\mathbf{v}, \mathbf{f}_v\}$		
$h = v^2/2 - \mu/r$	=	integral of energy
I	=	inclination
$\mathbf{I}$	=	unit matrix of third or sixth order
M	=	mean motion
n	=	number of time subintervals
p	=	semi-latus rectum
$\mathbf{p} = \{\mathbf{p}_r, \mathbf{p}_v\}$	=	vector of adjoint variables
$\mathbf{p}_v$	=	Lawden's primer vector
$\mathbf{q} = \mathbf{q}(\mathbf{q}_0, t)$	=	m-dimensional vector of instantaneous orbital elements, $m \le 5$
$\mathbf{q}_i$, $\mathbf{q}_f$	=	vectors of instantaneous elements of the initial and final orbits
$\mathbf{r}$	=	position vector
t	=	time

$t_1, \ldots, t_{n-1}$	=	instants dividing the subintervals
$\mathbf{v}$	=	velocity vector
$W_e = W_e(\mathbf{r}, t)$	=	effective electric power
$\mathbf{x} = \mathbf{x}(t)$	=	state vector of the transfer trajectory
$\mathbf{y}_i = \mathbf{y}_i(t)$, $\mathbf{y}_f = \mathbf{y}_f(t)$	=	state vectors of the initial and final orbits
$\mathbf{y}_j = \mathbf{y}_j(t)$	=	state vector in the jth reference orbit
$\boldsymbol{\alpha}$	=	jet acceleration vector
$\alpha = \lvert\boldsymbol{\alpha}\rvert$	=	acceleration value
$\Delta\mathbf{q} = \mathbf{q}_f - \mathbf{q}_i$		
ϖ	=	argument of periapsis
Ω	=	longitude of the ascending node

Subscripts “0”, “T” denote values of parameters at instants 0, T.
Subscript “j” denotes values of parameters in jth reference orbit ($j = 1, \ldots, n$).
Superscript “t” denotes transposition.

1. Introduction

Low thrust power-limited transfers between two given orbits are considered in this paper. The only requirement to the force field is that the motion could be given by instantaneous orbital elements. An emphasis is laid on spiral transfers with a high number of orbits what makes transfer optimization difficult. This case takes place for example for the motion in a strong gravity field of the central attracting body where the jet acceleration is much smaller than the gravity acceleration. There are various methods for the multi-revolution transfer optimization [1–7]. However, most of the methods are rather complicated [3, 5] or have a limited application (in particular, are applicable only to circular or neighboring orbits [1, 6, 7] or to coaxial or coplanar orbits [2, 4]).

A simple method for obtaining considered spiral transfer trajectory is suggested in this paper. The method is based on linearization of the motion near a set of short arcs of reference orbits. In this respect the method is similar to the modified method of transporting trajectory (MTT) [8–11] calculating power-limited transfers between two given positions. The main difference is that the instantaneous orbital elements of the reference orbits are taken as independent variables in the suggested method, whereas the MTT uses state vectors as independent variables. This is why the method described in this paper, contrary to the MTT, can be applied to the multi-revolution transfers between two given orbits. Any required accuracy may be reached using a suitable number of the reference orbits.

The suggested method makes it possible obtaining three types of transfers, such as follows:

– transfer from a given state vector to a given orbit with obtaining an optimal position of the entry into the final orbit;

– transfer from a given orbit to a given state vector with obtaining an optimal position of the start from the initial orbit;

– transfer between two given orbits with obtaining optimal initial and final positions of the transfer trajectory.

All of the three transfer types are considered in the paper. The suggested method also is applicable for the case of partly given final orbit; for example, when only energy or semi-major axis and eccentricity of the final orbit are specified. Like MTT, this method can be used if a constraint on the thrust direction is given [12, 13].

The linear approach used in the method does not make it possible obtaining global optimum of the solution; it is optimal in each point of the transfer trajectory. Thus, the method may be called as locally optimal. This deficiency of the method is compensated by its simplicity and effectiveness: the method does not need any averaging used by most of the existing methods and works well in the case of a big difference between initial and final orbits. Numerical examples demonstrating high effectiveness and wide applicability of the suggested method are also given.

2. Formulation of the Problem

Equation of motion subject to low thrust is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g} \tag{1}$$

where

$$\mathbf{g} = \{\mathbf{0}, \boldsymbol{\alpha}\} \tag{2}$$

The performance index for the LP propulsion is

$$J = \int_0^T \frac{\alpha^2}{2W_e} dt \tag{3}$$

Minimum value of the performance index gives minimum propellant consumption.

The problem is to find the thrust vector transferring the spacecraft between two orbits given by elements $\mathbf{q}_i, \mathbf{q}_f$ in time T and minimizing the performance index (see Fig. 1).

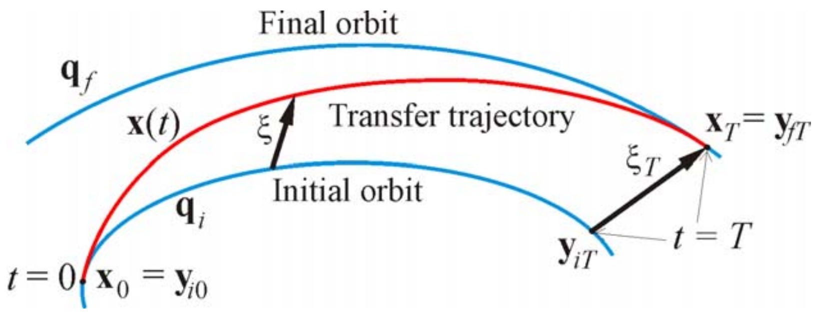

Figure 1. Transfer between given orbits

The boundary values of the problem are

$$\mathbf{x}_0 = \mathbf{y}_{i0}, \quad \mathbf{x}_T = \mathbf{y}_{fT}. \tag{4}$$

3. Transfer from a Given State Vector to a Given Orbit

Let us first assume that in Eq. (4) $\mathbf{y}_{i0}$ is given and $\mathbf{y}_{fT}$ is not given.

3.1. Neighboring Orbits

Let us consider low-thrust transfer between two neighboring orbits and introduce vector

$$\xi = \xi(t) = \mathbf{x}(t) - \mathbf{y}_i(t) \tag{5}$$

Due to the closeness of the initial and final orbits the equation of motion can be linearized near the initial orbit as follows

$$\dot{\boldsymbol{\xi}} = \mathbf{F}\boldsymbol{\xi} + \mathbf{g} \tag{6}$$

where

$$\mathbf{F} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}_i} \tag{7}$$

(vector **f** and respectively matrix **F** in Eqs. (6, 7) are calculated in the initial orbit). Note that Eq. (6) is not autonomous because matrix **F** is a function of time. The Hamiltonian for Eq. (5) is

$$H = -\frac{\alpha^2}{2W_e} + \mathbf{p}^t\mathbf{F}\boldsymbol{\xi} + \mathbf{p}_v^t\boldsymbol{\alpha} + p_t \tag{8}$$

where $\mathbf{p} = \{\mathbf{p}_r, \mathbf{p}_v\}$ is a vector of costate variables, $\mathbf{p}_v$ is Lawden's primer vector, p_t is a costate variable corresponding to additional equation $\dot{t} = 1$ making the system autonomous. Vector **p** satisfies the costate variational equation

$$\dot{\mathbf{p}}^t = -\left(\frac{\partial H}{\partial \boldsymbol{\xi}}\right)^t = -\mathbf{p}^t\mathbf{F} \tag{9}$$

Let the 6-order matrix $\boldsymbol{\Psi} = \boldsymbol{\Psi}(t)$ be a general solution to Eq. (9) with initial value $\boldsymbol{\Psi}_0 = \mathbf{I}$. Matrix $\boldsymbol{\Psi}$ can be represented by

$$\boldsymbol{\Psi} = \left[\boldsymbol{\Psi}_r \ \boldsymbol{\Psi}_v\right] \tag{10}$$

where $\boldsymbol{\Psi}_r, \boldsymbol{\Psi}_v$ are 6×3-dimensional sub-matrices. Then, the costate variables can be represented as follows

$$\mathbf{p} = \boldsymbol{\Psi}^t\boldsymbol{\beta}, \quad \mathbf{p}_v = \boldsymbol{\Psi}_v^t\boldsymbol{\beta} \tag{11}$$

where $\boldsymbol{\beta}$ is a constant vector. Function (8) reaches its maximum if

$$\boldsymbol{\alpha} = W_e\mathbf{p}_v = W_e\boldsymbol{\Psi}_v^t\boldsymbol{\beta} \tag{12}$$

Solution to Eq. (6) is given by the Cauchy formula

$$\boldsymbol{\xi} = \boldsymbol{\xi}(t) = \boldsymbol{\xi}_0 + \int_0^t \boldsymbol{\Phi}(t,\tau)\mathbf{g}\, d\tau \tag{13}$$

where $\boldsymbol{\Phi}(t_1, t_2) = \partial\mathbf{y}_i(t_1)/\partial\mathbf{y}_i(t_2)$ is the state transition matrix. Due to Eqs. (4, 5) $\boldsymbol{\xi}_0 = \mathbf{0}$ in Eq. (13). Using equations

$$\boldsymbol{\Phi}(t,\tau) = \boldsymbol{\Phi}(t,0)\boldsymbol{\Phi}(0,\tau) = \boldsymbol{\Phi}(t,0)\boldsymbol{\Phi}^{-1}(\tau,0), \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}(t,0) = \boldsymbol{\Psi}^{-1} \tag{14}$$

and Eqs. (2, 11, 12) solution (13) may be represented in the form

$$\boldsymbol{\xi} = \boldsymbol{\Phi}\mathbf{S}\boldsymbol{\beta} \tag{15}$$

where

$$\mathbf{S} = \mathbf{S}(t) = \int_0^t W_e \mathbf{\Psi}_v \mathbf{\Psi}_v^t dt \tag{16}$$

is a 6-order symmetric matrix. Now in order to find optimal thrust vector and state vector given by (12, 15) it is sufficient to obtain vector $\boldsymbol{\beta}$.

Since the final state vector $\mathbf{y}_{fT}$ of the transfer is not given it can be found from the transversality condition which in the considered case is

$$\mathbf{p}_T = \mathbf{p}(T) = \left(\frac{\partial \mathbf{q}_{fT}}{\partial \mathbf{y}_{fT}} \right)^t \boldsymbol{\sigma} \tag{17}$$

where $\boldsymbol{\sigma}$ is an arbitrary 5-dimensional constant vector. Due to the nearness of the initial and final orbits it can be taken $\partial \mathbf{q}_f / \partial \mathbf{y}_f \approx \mathbf{U}$ where

$$\mathbf{U} = \mathbf{U}(t) = \frac{\partial \mathbf{q}_i}{\partial \mathbf{y}_i}, \quad \mathbf{U}_0 = \mathbf{U}(0), \quad \mathbf{U}_T = \mathbf{U}(T) \tag{18}$$

is 5×6-dimensional matrix. Then the transversality condition (17) may be written as follows

$$\mathbf{p}_T = \mathbf{U}_T^t \boldsymbol{\sigma} \tag{19}$$

Equations (11, 14, 19) give

$$\boldsymbol{\beta} = \left(\mathbf{U}_T \mathbf{\Phi}_T \right)^t \boldsymbol{\sigma} \tag{20}$$

The following linear equation will be used:

$$\Delta \mathbf{q}_T = \mathbf{q}_{fT} - \mathbf{q}_{iT} = \mathbf{U}_T \boldsymbol{\xi}_T \tag{21}$$

On the other hand

$$\Delta \mathbf{q}_T = \mathbf{Q} \Delta \mathbf{q}_0, \quad \Delta \mathbf{q}_0 = \Delta \mathbf{q}(0) \tag{22}$$

where

$$\mathbf{Q} = \frac{\partial \mathbf{q}_{iT}}{\partial \mathbf{q}_{i0}} \tag{23}$$

Substituting Eqs. (15, 20) into Eq. (21) and using relations

$$\mathbf{Q}^{-1} \mathbf{U}_T \mathbf{\Phi}_T = \frac{\partial \mathbf{q}_{i0}}{\partial \mathbf{q}_{iT}} \frac{\partial \mathbf{q}_{iT}}{\partial \mathbf{y}_{iT}} \frac{\partial \mathbf{y}_{iT}}{\partial \mathbf{y}_{i0}} = \frac{\partial \mathbf{q}_{i0}}{\partial \mathbf{y}_{i0}} = \mathbf{U}_0 \tag{24}$$

equation (22) gives

$$\Delta \mathbf{q}_0 = \mathbf{Q}^{-1} \mathbf{U}_T \mathbf{\Phi}_T \mathbf{S}_T \left(\mathbf{U}_T \mathbf{\Phi}_T \right)^t \boldsymbol{\sigma} = \mathbf{W} \mathbf{Q}^t \boldsymbol{\sigma} \tag{25}$$

where

$$\mathbf{W} = \mathbf{U}_0\mathbf{S}_T\mathbf{U}_0^t \tag{26}$$

is a 5th order matrix. Then $\boldsymbol{\sigma} = \mathbf{Q}^{t^{-1}}\mathbf{W}^{-1}\Delta\mathbf{q}_0$ and vector $\boldsymbol{\beta}$ can be found from Eqs. (20, 24) as follows

$$\boldsymbol{\beta} = \mathbf{U}_0^t\mathbf{W}^{-1}\Delta\mathbf{q}_0 \tag{27}$$

Due to Eqs. (12, 15, 5, 27) the thrust vector and the state vector in the optimal transfer become

$$\boldsymbol{\alpha} = W_e\boldsymbol{\Psi}_v^t\mathbf{U}_0^t\mathbf{W}^{-1}\Delta\mathbf{q}_0 \tag{28}$$

$$\mathbf{x} = \mathbf{y}_i + \boldsymbol{\Phi}\mathbf{S}\mathbf{U}_0^t\mathbf{W}^{-1}\Delta\mathbf{q}_0 \tag{29}$$

Equations (4, 29) give the state vector of the entry into the final orbit as follows

$$\mathbf{y}_{fT} = \mathbf{y}_{iT} + \boldsymbol{\Phi}_T\mathbf{S}_T\mathbf{U}_0^t\mathbf{W}^{-1}\Delta\mathbf{q}_0 \tag{30}$$

Putting Eq. (28) into Eq. (3) and using Eqs. (16, 26) the performance index can be found as follows

$$J = \frac{1}{2}\Delta\mathbf{q}_0^t\mathbf{W}^{-1}\Delta\mathbf{q}_0 \tag{31}$$

3.2. Arbitrary Orbits

Here the low-thrust transfer between two arbitrary orbits is considered. Let us divide the time interval T into n subintervals defined by instants $t_0 = 0, t_1, ..., t_{n-1}, t_n = T$; also let us assume that $n - 1$ intermediate reference orbits between the initial and final orbits are specified somehow and $\mathbf{q}_1(t), ..., \mathbf{q}_{n-1}(t)$ are 5-dimentional vectors of elements of the reference orbits (see Fig. 2). These elements may be given, for instance, in the following way:

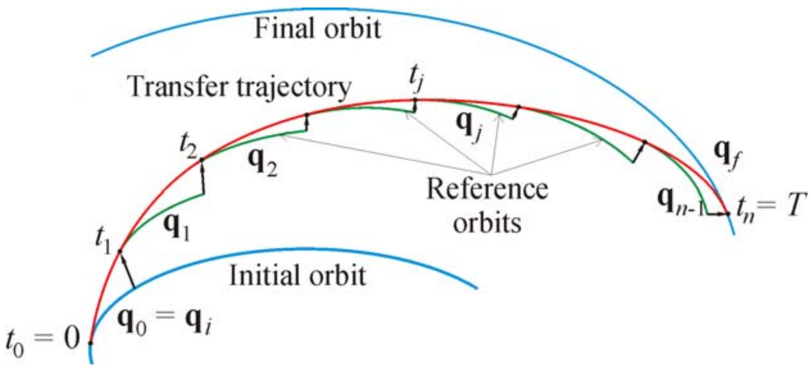

Figure 2. Transfer between arbitrary orbits

$$\mathbf{q}_j(0) = \mathbf{q}_i(0) + \frac{j}{n}\left(\mathbf{q}_f(0) - \mathbf{q}_i(0)\right), \quad j = 1, ..., n-1 \tag{32}$$

Designating $\Delta\mathbf{q}_j = \mathbf{q}_j(0) - \mathbf{q}_{j-1}(0)$, $j = 1, ..., n$ with $\mathbf{q}_0 = \mathbf{q}_i$, $\mathbf{q}_n = \mathbf{q}_f$, the following obvious equality is fulfilled:

$$\sum_{j=1}^{n}\Delta\mathbf{q}_j = \Delta\mathbf{q}_0 \tag{33}$$

Dividing the transfer trajectory into n arcs let us assume that the jth arc begins in the j–1st orbit and ends in the jth one. Also let us assume the number n big enough to make the j–1st and jth orbits close to each other for all j = 1,..., n. Then the results of the previous section may be applied to each of the arcs with linearization of motion in the jth transfer arc near the j–1st orbit. The problem is to find the reference orbits giving optimal transfer trajectory.

Due to Eq. (31) the performance index for the jth arc is

$$J_j = \frac{1}{2}\Delta\mathbf{q}_j^t \mathbf{W}_j^{-1}\Delta\mathbf{q}_j \tag{34}$$

where, similarly to Eqs. (26, 16, 18)

$$\mathbf{W}_j = \mathbf{U}_j\mathbf{S}_j\mathbf{U}_j^t, \quad \mathbf{S}_j = \mathbf{S}_j\left(t_j\right) = \int_{t_{j-1}}^{t_j} W_e \boldsymbol{\Psi}_{jv}\boldsymbol{\Psi}_{jv}^t dt, \quad \mathbf{U}_j = \frac{\partial\mathbf{q}_{j-1}(0)}{\partial\mathbf{y}_{j-1}(0)}, \tag{35}$$

$\mathbf{y}_j$ is state vector of the jth reference orbit, matrix $\boldsymbol{\Psi}_{jv}$ is calculated in the j–1st reference orbit. Performance index of the whole problem is

$$J = \sum_{j=1}^{n} J_j \tag{36}$$

In order to find the transfer trajectory that gives the minimum value of J, it is sufficient to find intermediate reference orbits that provide a minimum for Eq. (36). Thus, function (36) should be minimized with respect to the vectors $\Delta\mathbf{q}_j$, $j = 1,...,n$ taking into account Eq. (33). Let us introduce the helping function

$$L = J - \boldsymbol{\lambda}^t\left(\sum_{j=1}^{n}\Delta\mathbf{q}_j - \Delta\mathbf{q}_0\right)$$

where $\boldsymbol{\lambda}$ is a Lagrange multiplier. Necessary conditions of a minimum of the functional (36) are

$$\left(\frac{\partial L}{\partial\Delta\mathbf{q}_j}\right)^t = \mathbf{W}_j^{-1}\Delta\mathbf{q}_j - \boldsymbol{\lambda} = \mathbf{0}, \quad j = 1,...,n \tag{37}$$

Thus,

$$\Delta\mathbf{q}_j = \mathbf{W}_j\boldsymbol{\lambda}, \quad j = 1,...,n \tag{38}$$

and Eqs. (33, 38) give

$$\boldsymbol{\lambda} = \boldsymbol{\Omega}^{-1}\Delta\mathbf{q}_0 \tag{39}$$

where

$$\boldsymbol{\Omega} = \sum_{j=1}^{n}\mathbf{W}_j$$

Multiplying Eq. (37) by $\Delta\mathbf{q}_j$ and summing for all j = 1, …, n Eq. (36) can be transformed, using Eq. (34), to

$$J = \frac{1}{2}\boldsymbol{\lambda}^t \Delta\mathbf{q}_0 = \frac{1}{2}\Delta\mathbf{q}_0^t \boldsymbol{\Omega}^{-1}\Delta\mathbf{q}_0 \tag{40}$$

Optimal thrust vector and the state vector of the transfer trajectory in the *j*th subinterval are

$$\boldsymbol{\alpha}(t) = N\boldsymbol{\Psi}_{jv}^t(t)\mathbf{U}_j^t \boldsymbol{\Omega}^{-1}\Delta\mathbf{q}_0, \tag{41}$$

$$\mathbf{x}(t) = \mathbf{y}_{j-1}(t) + \boldsymbol{\Phi}_j(t,0)\mathbf{S}_j\mathbf{U}_j^t \boldsymbol{\Omega}^{-1}\Delta\mathbf{q}_0 \tag{42}$$

where $t_{j-1} \le t \le t_j$, $j = 1,..,n$.

3.3. Calculation Procedure

Let $\mathbf{y}_j^0, \mathbf{y}_j^1$ be the state vectors of the *j*th reference orbit at the beginning and at the end of the *j*th time subinterval (i.e. at times t_{j-1}, t_j respectively). Then, the solution to the problem considered here may be obtained by means of the following iterative calculation procedure:

1. $n - 1$ intermediate reference orbits are specified somehow, for example, using Eq. (32). A launch position in the initial orbit is specified (i.e. state $\mathbf{y}_0^0 = \mathbf{y}_i(0)$ is given) and the respective initial state vector of the transfer trajectory is $\mathbf{x}_0 = \mathbf{y}_0^0$.

2. Vector $\mathbf{y}_j^0$ is calculated for $j = 1$ using the following equation, similar to Eq. (30):

$$\mathbf{y}_j^0 = \mathbf{y}_{j-1}^1 + \boldsymbol{\Psi}_j^{-1}\mathbf{S}_j\mathbf{U}_j^t\mathbf{W}_j^{-1}\Delta\mathbf{q}_j \tag{43}$$

   where matrix $\boldsymbol{\Psi}_j = \boldsymbol{\Psi}(t_j)$ is calculated in the j – 1st reference orbit and matrices $\mathbf{S}_j$, $\mathbf{U}_j$ are given by Eq. (35). Since Eq. (43) is approximate, vectors $\mathbf{q}_j, \Delta\mathbf{q}_j$ should be recalculated as $\mathbf{q}_j = \mathbf{q}\left(\mathbf{y}_j^0\right)$, $\Delta\mathbf{q}_j = \mathbf{q}_j - \mathbf{q}_{j-1}$.

3. Step 2 is repeated for $j = 2, \ldots, n - 1$.

4. Vector (39) is found and new vectors $\Delta\mathbf{q}_j$ are calculated using Eq. (38). Then new reference orbits with elements $\mathbf{q}_{j+1} = \mathbf{q}_j + \Delta\mathbf{q}_i$ $(j = 0,...,n-1)$ are determined.

5. Performance index is calculated using Eq. (40) and steps 2–4 are repeated until decrement ΔJ of the performance index gets smaller than a given parameter $\varepsilon > 0$. As soon as $|\Delta J| < \varepsilon$ the thrust vector $\boldsymbol{\alpha}$ and the state vector $\mathbf{x}$ of the transfer trajectory may be calculated at each time subinterval using Eqs. (41, 42).

The suggested method is approximate, although any desired accuracy may be reached by means of selecting an appropriate amount *n* of subintervals.

4. Other Transfer Types

4.1. Transfer from a Given Orbit to a Given State Vector

Now let us consider the case when the launch position can be selected in the initial orbit arbitrarily (i.e. state vector $\mathbf{y}_{i0}$ is not given in Eq. (4)) and the position of the entry into the final orbit is given (i.e. vector $\mathbf{y}_{fT}$ is given in Eq. (4)). This case takes place, for example, for a transfer from any point situated in a low Earth orbit to a specified geostationary position. In this case the method described in Section 3 should be applied in the backward direction with retrograde time, i.e. vector $\mathbf{y}_{n-1}^{0}$ of the start from n – 1st reference orbit can be found for a given state vector $\mathbf{y}_{fT}$ of the arrival to the final orbit etc., until vector $\mathbf{y}_{i0}$ is found. The equations to solve the problem considered here can be easily derived from the equations given in Section 3.

4.2. Transfer between Two Given Orbits

This is the classical case of interorbital transfer when the optimal launch and arrival positions in the initial and final orbits are to be found. This case may be solved using the suggested method in the following way:

A first guess for the launch position should be given somehow. This position defines vector $\mathbf{y}_{i0}$ and, in the first iteration of the calculation procedure described in subsection 3.3 of section 3, the final state vector $\mathbf{y}_{fT}$ may be found. In the second iteration of the calculation procedure for this state vector a new value of the vector $\mathbf{y}_{i0}$ may be found as described in section 4.1 etc., i.e. odd iterations of the calculation procedure use the case described in section 3 and even iterations use the case described in subsection 4.1.

5. Partly Given Final Orbit

The suggested method also can be used in the case of partly given elements of the final orbit, i.e. if vector $\mathbf{q}_f$ has dimension $m < 5$. For instance, only energy of the final orbit ($m = 1$) or semi-major axis and eccentricity ($m = 2$) may be given. In this case vectors $\mathbf{q}_j$ of the elements of the intermediate reference orbits also are m-dimensional with the same orbital elements as $\mathbf{q}_f$. Matrices $\mathbf{U}_j$ and $\mathbf{W}_j$ in Eq. (35) have dimension $m\times6$ and order m respectively. Non-given orbital elements are determined by means of the transversality condition (17) and the respective conditions for vectors $\mathbf{q}_j$.

6. Constrained Thrust Direction

Let us assume that there is a constraint on the thrust vector $\boldsymbol{\alpha}$ given by

$$\mathbf{B}\boldsymbol{\alpha} = \mathbf{0} \tag{44}$$

where $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is a matrix of dimension 1×3 ($\mathbf{B}$ is a row) or 2×3 (i.e. the thrust direction is given). In this case, the suggested optimization method is also applicable with matrix $\mathbf{S}_j$ and vector $\boldsymbol{\alpha}_j$ from Eqs. (35, 41) replaced by

$$\mathbf{S}_j = \mathbf{S}_j(t) = \int_{t_j}^{t} W_e \boldsymbol{\Psi}_{jv} \mathbf{P} \boldsymbol{\Psi}_{jv}^t dt, \quad \boldsymbol{\alpha}_j = W_e \mathbf{P} \boldsymbol{\Psi}_{jv}^t \mathbf{U}_j^t \mathbf{W}_j^{-1} \Delta\mathbf{q}_j$$

where third-order matrix

$$\mathbf{P} = \mathbf{I} - \mathbf{B}^t \left(\mathbf{B}\mathbf{B}^t\right)^{-1} \mathbf{B} \tag{45}$$

projects any vector onto the constraining set given by Eq. (44) (projective matrix) [13, 14].

7. Local Optimality of the Method

The solution to the transfer problem described in section 3 is not globally optimal. It will be shown in this section that the solution gives an optimal direction of the thrust vector at each point of the transfer trajectory, i.e. gives a local optimum. Neighbouring initial and final orbits will be considered for simplicity.

Let us consider the case when a given value of a unique orbital element is to be reached, i.e. $m = 1$ and $\mathbf{q} = q$ is a given scalar element. Thus, Δq_0 is a scalar as well. Then $\mathbf{U}_0$ given by Eq. (18) is a row and $\mathbf{W}$ in Eq (26) is a scalar. Then, equation (28) becomes

$$\boldsymbol{\alpha} = W_e \boldsymbol{\Psi}_v^t \mathbf{U}_0^t \mathbf{W}^{-1} \Delta q_0 = s\left(\mathbf{U}_0 \boldsymbol{\Psi}_v\right)^t = s\left(\frac{\partial q_i}{\partial \mathbf{y}_{i0}} \frac{\partial \mathbf{y}_{i0}}{\partial \mathbf{v}_i}\right)^t = s\left(\frac{\partial q_i}{\partial \mathbf{v}_i}\right)^t \tag{46}$$

where $s = W_e \Delta q_0 / \mathbf{W}$ is a scalar parameter, $\mathbf{v}_i$ is the velocity vector in the initial orbit. An instantaneous change of the parameter q by means of the thrust during a small time interval Δt is

$$\Delta q = \frac{\partial q}{\partial \mathbf{v}} \boldsymbol{\alpha} \Delta t \approx \frac{\partial q_i}{\partial \mathbf{v}_i} \boldsymbol{\alpha} \Delta t \tag{47}$$

Thus, maximum value of Δq is reached when vector $\boldsymbol{\alpha}$ in Eq. (47) is directed along $\partial q_i / \partial \mathbf{v}_i$. That means that vector (46) is optimal at time t, i.e. the suggested method gives locally optimal solution.

8. Special Cases of the Gravity Field

8.1. Transfers near an Oblate Planet

A motion near an oblate planet is considered here under the assumption that there are no more perturbations. Taking into account only secular perturbations, five orbital elements defining orbit are

$$a(t) = a(0), \quad e(t) = e(0), \quad i(t) = i(0),$$

$$\Omega(t) = \Omega(0) - \frac{3}{2} J_2 n \left(\frac{R_e}{p}\right)^2 \cos i \, t, \quad \omega(t) = \omega(0) + \frac{3}{4} J_2 n \left(\frac{R_e}{p}\right)^2 \left(5\cos^2 i - 1\right) t$$

where R_e, J_2, n are equatorial radius of the planet, coefficient of the second zonal harmonic, and mean motion respectively. Mean anomaly necessary for calculation of motion in the reference orbits is

$$M = M_0 + \left(\frac{3}{4} J_2 \left(\frac{R_e}{p}\right)^2 \sqrt{1-e^2} \left(3\cos^2 i - 1\right) + 1\right) nt$$

where $M_0 = M(0)$.

8.2. Two Body Problem

In the two body problem initial, final and reference orbits are Keplerian ones with state vectors $\mathbf{y} = \mathbf{y}(t)$ and

$$\mathbf{q}(t) = \mathbf{q}(0) = \mathbf{q}_0 \tag{48}$$

(subscripts *i* or *j* are skipped here and below for simplicity). The orbital elements are first integrals of the motion. Assuming vector $\mathbf{q}$ *m*-dimensional (i.e. $\mathbf{q} = \{q_1, ..., q_m\},\ m \le 5$) let us consider an extended vector of orbital elements

$$\tilde{\mathbf{q}} = \{q_1, ..., q_6\} = \{\mathbf{q}, q_{m+1}, ..., q_6\}$$

Besides matrix $\mathbf{U}$ given by Eq. (18) let us consider extended matrix

$$\tilde{\mathbf{U}}(t) = \frac{\partial \tilde{\mathbf{q}}}{\partial \mathbf{y}}$$

The state and costate transition matrices can be represented in the form [15, 16]

$$\mathbf{\Phi} = \frac{\partial \mathbf{y}}{\partial \mathbf{y}_0} = \tilde{\mathbf{U}}^{-1}\tilde{\mathbf{U}}_0, \quad \mathbf{\Psi} = \mathbf{\Phi}^{-1} = \tilde{\mathbf{U}}_0^{-1}\tilde{\mathbf{U}}$$

Using Eqs. (16, 18, 48) and equation $\mathbf{\Psi}_v = \partial \mathbf{y}_0 / \partial \mathbf{v}$ matrix (26) may be found as follows

$$\mathbf{W} = \frac{\partial \mathbf{q}}{\partial \mathbf{y}_0} \int_0^t W_e \frac{\partial \mathbf{y}_0}{\partial \mathbf{v}} \left(\frac{\partial \mathbf{y}_0}{\partial \mathbf{v}} \right)^t d\tau \left(\frac{\partial \mathbf{q}}{\partial \mathbf{y}_0} \right)^t = \int_0^t W_e \frac{\partial \mathbf{q}}{\partial \mathbf{v}} \left(\frac{\partial \mathbf{q}}{\partial \mathbf{v}} \right)^t d\tau \tag{49}$$

Matrices $\tilde{\mathbf{U}}, \tilde{\mathbf{U}}^{-1}, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{W}$ in the two body problem are calculated in [15, 16] analytically[1]. Thus, the suggested method is analytical in the case of the two body problem.

9. Numerical Examples

This section illustrates the suggested method by examples of transfers in the Earth's sphere of influence. Two body problem is considered. Orbits are given by the orbital elements

$$\mathbf{q} = \{r_\pi, r_\alpha, i, \Omega, \omega\}$$

where r_π, r_α are radii of perigee and apogee in thousands of kilometers (Mm), *i* is the orbital inclination, Ω is the longitude of the ascending node, ω is the argument of perigee. Angular elements are given in degree. The transfer time and number of subintervals are equal to $T = 400$ hour and $n = 5000$ in all examples considered below.

9.1. Transfer between Elliptical Orbits with High Mutual Inclination

Transfer between two orbits given by

$$\mathbf{q}_i = \{7, 30, 50, 80, -60\},\ \mathbf{q}_f = \{40, 80, 80, -80, 70\}$$

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[1] Matrices $\tilde{\mathbf{U}}$ and **W** are named as **A** and **S** respectively in [15, 16].

is considered. The transfer trajectory is shown in Fig. 3 in two projections onto the equator plane *xy* and the polar plane *xz*. The jet acceleration value divided by g = 9.8066 m/s^2 is shown in Fig. 4. Performance index and total ΔV for the transfer are J = 46.91 m^{2}s^{-3} and ΔV = 10.58 km/s.

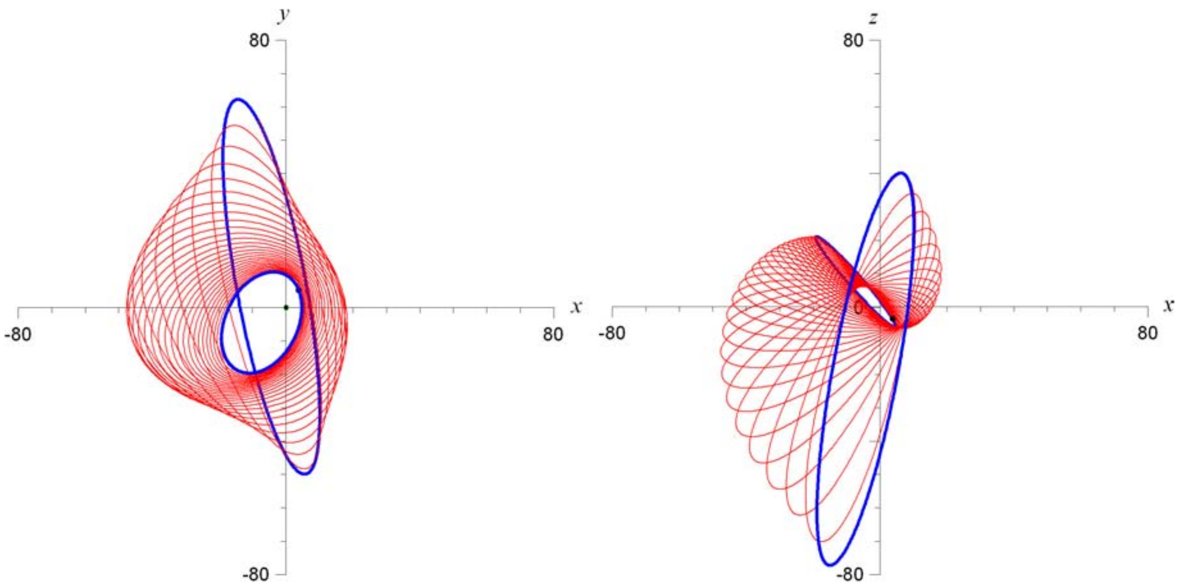


Figure 3. Transfer between two elliptic orbits with high mutual inclination

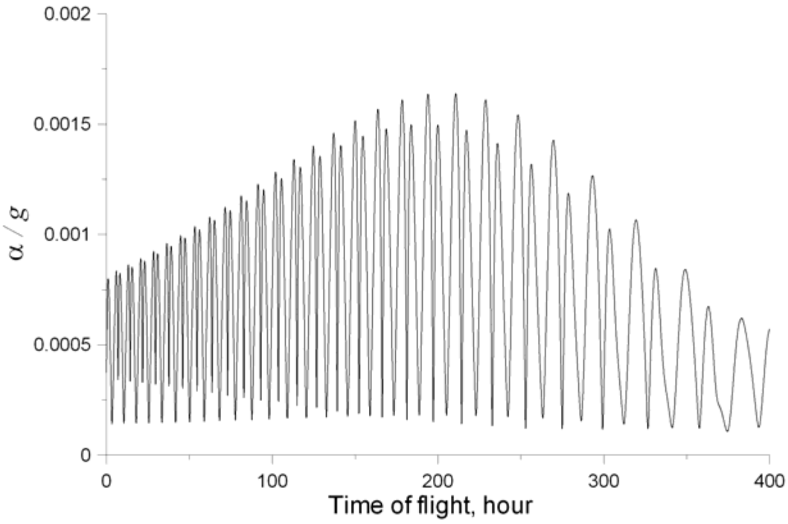


Figure 4. Acceleration value for the transfer between two elliptic orbits with high mutual inclination

9.2. Transfer to an Orbit with Given Perigee and Apogee Radii

Transfer to a partly given orbit, namely to an orbit with given only perigee and apogee radii, is considered here. Optimal transfer is planar in this case. Only perigee and apogee radii of the initial orbit are specified, because the other initial orbital elements are not important and may be taken equal to zero. Elements of the initial and final orbits are taken as follows:

$$\mathbf{q}_i = \{7, 20\},\ \mathbf{q}_f = \{40, 80\}$$

Transfer orbit is shown in Fig. 5 and the respective propulsion acceleration value is given in Fig. 6. Performance index and total ΔV are J = 3.02 m^{2}s^{-3} and ΔV = 2.81 km/s. As is seen in Fig. 5, the optimal attitude of the final orbit is coaxial with the initial one.

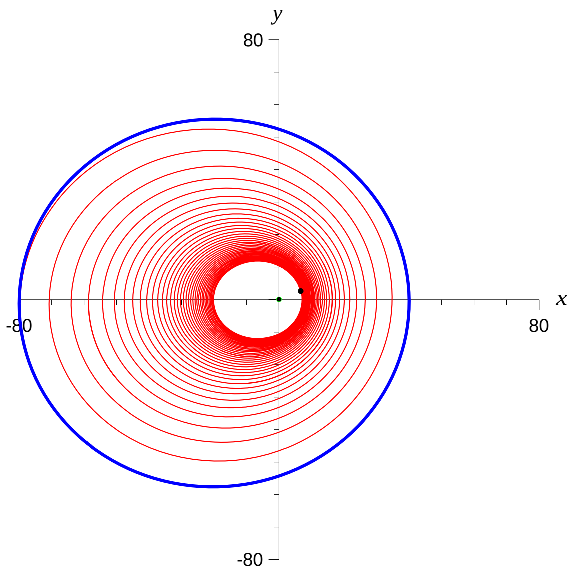


Figure 5. Transfer trajectory to an orbit with given perigee and apogee radii

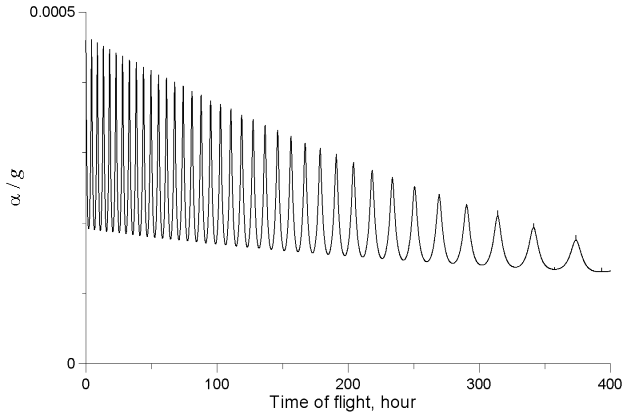


Figure 6. Acceleration value for the transfer to an orbit with given perigee and apogee radii

9.3. Constrained Thrust Direction

A transfer between the orbits given by the elements

$$\mathbf{q}_i = \{7, 20, 0, 0, 0\},\ \mathbf{q}_f = \{40, 80, 30, 60, 60\},$$

is considered here. The transfer trajectory is shown in Fig. 7. The performance index and total ΔV are J = 5.77 m^2s^{-3} and ΔV = 3.94 km/s. Now let us assume the thrust is always orthogonal to the spacecraft position vector, i.e. $\mathbf{B} = \mathbf{r}^t$ in Eq. (44). The projective matrix (45) in this case is

$$\mathbf{P} = \mathbf{I} - \frac{\mathbf{r}\mathbf{r}^t}{r^2}$$

The transfer trajectory for the constrained thrust direction visually does not differ from the one for the unconstrained direction shown in Fig. 7. Performance index and total ΔV in the case of the constrained thrust direction are J = 9.47 m^2s^{-3} and ΔV = 4.53 km/s. Acceleration value versus time for the unconstrained and constrained thrust direction is shown in Fig. 8.

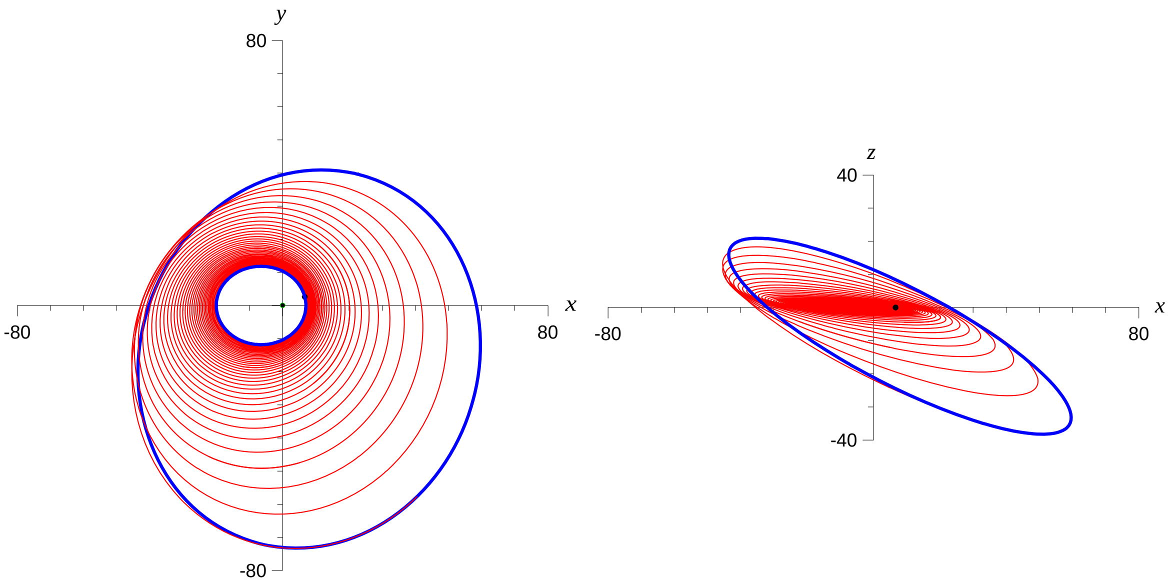
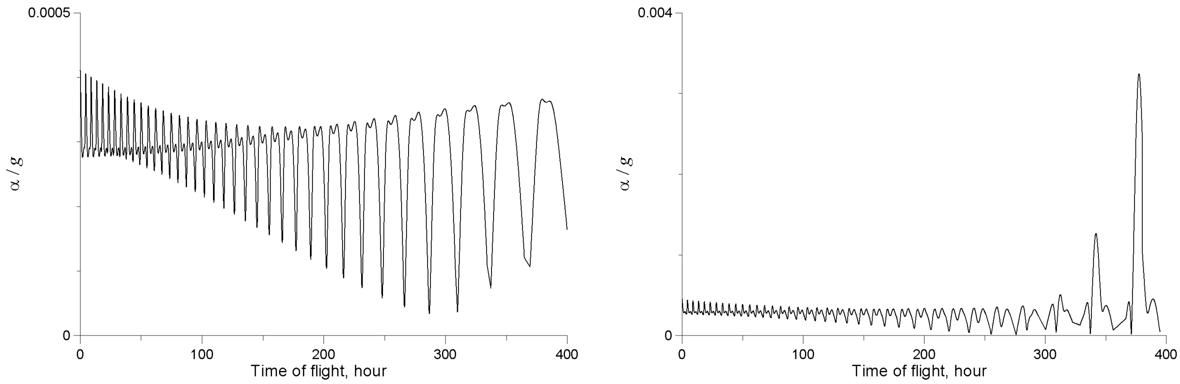


Figure 7. Transfer trajectory for the unconstrained thrust direction



a) **b)**

Figure 8. Acceleration values for the unconstrained (a) and the constrained (b) thrust direction

10. Conclusions

The suggested method has two disadvantages: it does not give a globally optimal solution and it is applicable only to the power-limited thrust, whereas the existing thrusters has characteristics close to the constant exhaust velocity. Although these disadvantages are compensated by the following advantages of the method: it is good for any gravity field and any number of orbits of the transfer trajectory. The method is semi-analytical in a general case, and analytical for the two body problem, what simplifies a qualitative analysis of the solution. The method is simple and fast; it can be applied to the state vector-to-orbit, orbit-to-state vector, and orbit-to-orbit transfers. Also the method is applicable to the cases of partly given final orbits and a constraint put on the thrust direction. The suggested method can be used at early phases of the mission design or for obtaining a first guess for an accurate solution.

11. Acknowledgement

The authors are grateful to the Sao Paulo Research Foundation FAPESP (Brazil) for the financial support of this study.

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