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The random spread model

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Abstract The paper proposes the new stochastic model of a Random Spread for describing the spatial propagation of sequential events such as forest fires. A Random Spread is double Markov chain, whose each step is a (random) set operator β combining a Cox process with a Boolean random closed set. Under iteration, operator β provides the time evolution of the Random Spread, which turns out to be a birth-and-death process. Average sizes, and the probabilities of extinction are derived. The random spread model was applied to the analysis of the fires that occurred in the State of Selangor (Malaysia) from 2000 to 2004. It was able to predict all places where burnt scars actually occurred, which is a strongly significant verification.

Keywords: Random Spread, Boolean RACS, Markov chain, birth-and-death, random closed sets, forest fires, simulation, mathematical morphology.

1. Introduction

In a large number of wild forests, such as typically in South East Asia, forest fires propagate less under the action of the wind, as in Mediterrenan countries, than under almost isotropic causes. For instance, the fires that occurred in the state of Selangor (Malaysia, see Figure 1) from 2000 to 2004 seem to evolve randomly: the initial seats vanish sometimes, but may also give birth to new seats at some distance. The foresters use to describe the fire progress by means two key maps [16], namely the daily *spread rate* of the fire, i.e., the radius r of the daily circular propagation of the fire, and the *fuel consumption* f_w, (a weighted version of the vegetation map). Both are depicted in Figure 1. Clearly, a straightforward use of such key maps does lead to a pertinent description. By starting from *any* point seat, one always arrives to burn the whole country in a finite time, under iteration, as both maps are positive. Indeed, one must use the rate information in some restrictive way, to be able to reach actual events. It is exactly the purpose of the present paper.

2. Stochastic models for growth

One can classify the proposed model as a discrete branching process which generalises the Galton-Watson process, since it involves a location of the new

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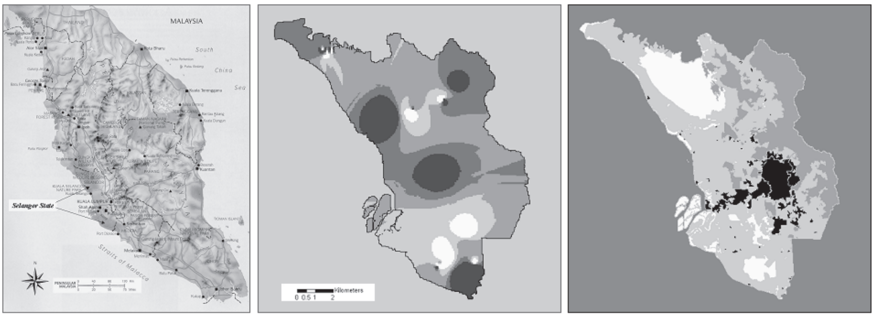

a) b) c)

Figure 1. (a) State of Selangor in Malaysia. (b) Map of the spread rate, i.e., of the radius r of the daily circular propagation of the fire. (c) Map f_w of the fuel consumption.

generation in space. Since C. J. Preston's pioneer work on *spatial birth-and-death* [10], one finds in the literature several point stochastic processes for describing joint evolutions in space and time. Their characteristic functionals $Q(K)$ are generally inaccessible, but they yield significant simulations. A second class is the concern of "thick" structures, i.e., which do not reduce to isolated points. A particular case can be found in [11], p.562, under the name of hierarchical RACS, with several variants such as the following: *the RACS at time t is X_t, and X_{t+dt} is generated by adding to X_t any boolean grain which occurs during $[t, t+dt]$ and whose center hits X_t.* In spite of its outward simplicity, this variant is not tractable, i.e., one cannot express the functional Q_t of X_t by means of the initial conditions, as proved by D. Jeulin in [5].

However, a number of phenomena, including forest fires, follows the same type of behavior. Each time that in the mineral, vegetal or animal worlds, seeds move and then develop a new colony, they involve some random sequential growth. But how to model it by tractable random closed set (RACS)? The trouble with the hierarchical RACS comes from that their evolution between steps n and $n+1$ refers to the *whole past*, from 0 to n. If we relax this condition, can we reach more tractable growth RACS? In addition, we must take into account that the space parameters which govern the evolution laws (e.g. the fuel amount for forest fires) usually vary from place to place, so that the new model should not be *a priori* translation invariant, but accept some imposed heterogeneity.

That are the questions we consider in this paper, by proposing the *Random Spread* RACS.

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3. Reminders

The random spread model in presented in the framework of the Euclidean space $\mathbb{R}^d$ of dimension d. Denote by the $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ (resp. $\mathcal{F}$, $\mathcal{K}$) the family of all sets (resp. closed sets, compact sets) of $\mathbb{R}^d$. Symbol $\mathcal{S}$ stands for the singletons of $\mathcal{P}(\mathbb{R}^d)$ and the same symbol, e.g. x, is used for the points of $\mathbb{R}^d$ and for the elements of $\mathcal{S}$.

Set dilation A *structuring function* is an arbitrary mapping $x \longmapsto \delta(x)$ from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{P}(\mathbb{R}^d)$. It extends into the *dilation* δ from $\mathcal{P}(\mathbb{R}^d)$ into itself [4, 7, 12] by the relation

$$\delta(X) = \bigcup \{\delta(x)\ ,\ x \in \mathcal{S}\} \qquad\qquad X \in \mathcal{P}(\mathbb{R}^d). \tag{1}$$

In the following, all structuring functions are supposed to be *compact*, i.e.,

Definition 1. A structuring function $\delta : \mathcal{S}(\mathbb{R}^d) \longmapsto \mathcal{P}(\mathbb{R}^d)$ is said to be *compact* when

i) it is upper semi continuous from $\mathcal{S}$ into $\mathcal{K}$,

ii) the union $\bigcup\{\delta_{-x}(x), x \in \mathbb{R}^d\}$ has a compact closure δ

$$\overline{\delta_0} = \overline{\bigcup\{\delta_{-x}(x), x \in \mathbb{R}^d\}}. \tag{2}$$

By extension, the associated dilation $\delta : \mathcal{P} \longmapsto \mathcal{P}$ is also said to be compact. □

The *reciprocal duality* between dilations plays an important role below. With each structuring function δ, associate the *reciprocal structuring function* ζ by writing

$$y \in \zeta(x) \qquad \text{if and only if} \qquad x \in \delta(y) \qquad\qquad x, y \in E. \tag{3}$$

The algorithm that expresses ζ in function of δ is therefore

$$\zeta(x) = \bigcup\{y : x \in \delta(y)\}. \tag{4}$$

Random Closed Sets (Euclidean case) The results which follow, on RACS and Boolean RACS, are basically due to G. Matheron [8]. Instructive introductions to RACS may be found in [9], or [15]. Given an element $K \in \mathcal{K}$, consider the class

$$\mathcal{F}^K = \{F : F \in \mathcal{F},\ F \cap K = \emptyset\}$$

of all closed sets that miss K.

As K spans the family $\mathcal{K}$, the classes $\left\{\mathcal{F}^K, K \in \mathcal{K}\right\}$ are sufficient to generate the σ-algebra. Moreover, as set $\mathcal{F}$ is compact for the hit-or-miss topology, there exist probabilities, Pr say, on σ_f, and each triplet $(\mathcal{F}, \sigma_f, \Pr)$ defines a RACS.

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Theorem 1. *The distribution of a RACS X is* uniquely *defined by the datum of the probabilities*

$$Q(K) = \Pr\{K \subseteq X^c\} \tag{5}$$

as K spans the class $\mathcal{K}$ of the compact sets of $\mathbb{R}^d$. Conversely, a family $\{Q(K),\, K \in \mathcal{K}\}$ defines a (necessarily unique) RACS if and only if $1 - Q(K) = T(K)$ is an alternating Choquet capacity of infinite order such that $0 \leq T \leq 1$ and $T(\emptyset) = 0$.

The mapping $Q : \mathcal{K} \rightarrow [0,1]$ is called the characteristic functional *of the RACS.*

The characteristic functional Q plays w.r.t. a random closed set X the same role as the distribution function for a random variable x. If $\{X_i, i \in I\}$ stands for all possible realizations of the RACS X, and if $\psi : \mathcal{F} \longmapsto \mathcal{F}$ is semi-continuous, hence measurable, then the family $\{\psi(X_i),\, i \in I\}$ characterizes in turn all realizations of a RACS, denoted by $\psi(X)$. This basic result allows us to play with RACS just as with deterministic sets, to intersect them, to dilate them, etc.

Boolean RACS The Boolean RACS, is very popular and led to many variants (e.g., [1, 6, 11, 15]). In the present study, we specify it as follows. Consider the two primitives of

i/ a Poisson process $\mathcal{J}(\theta)$, whose intensity measure θ is upperbounded, i.e., $\theta(dx) \leq \overline{\theta}.dx$ with $\overline{\theta} < \infty$

ii/ a compact, and deterministic, structuring function $\delta : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{K}(\mathbb{R}^d)$ called “primary grain”.

The Boolean RACS X is constructed in two steps. First, take a realization J of Poisson points, which provides the set of points x_j, $x_j \in J$. Second, take the union X of all primary grains whose centers belong to the Poisson realization, i.e.,

$$X = \bigcup\{\delta_{x_j}\, , x_j \in J\}.$$

This union generates a realization of the Boolean RACS X. The characteristic functional Q of RACS X derives easily form the above definition [8], and equals

$$Q(K) = \exp - \int \theta(dz) 1_{\delta(z) \cap K} = \exp - \int \theta(dz) 1_{z \cap \zeta(K)} = \exp -\theta[\zeta(K)]. \tag{6}$$

where ζ is the dilation reciprocal of δ. If we restrict the previous Poisson points to those which occur in a given compact set X_0, then this comes back to change the intensity $\theta(dx)$ into $\theta^*(dx) = \theta(dx).1_{X_0}(x)$, the Boolean structure being preserved, so that

$$\Pr\{K \subseteq X_1^c\} = Q_1(K) = e^{-\theta^*[\zeta(K)]} = e^{-\theta[\zeta(K) \cap X_0]}. \tag{7}$$

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4. Random Spreads

4.1 Definition

The Random Spread model generalizes Matheron's Boolean RACS by introducing a genetic dimension, namely the successive steps, according to which the $(n+1)^{th}$ Boolean RACS depends on the realization of the n^{th} one.

Consider an initial random seat I_0 made of an a.s. locally finite number of initial point seats in $\mathbb{R}^d$. The fire evolution from I_0 is the concern, on the one hand, of the fire the initial seats provoke, or *fire spread* $X_1 = \delta(I_0)$, and on the other hand of the generation of subsequent *seats spread* $I_1 = \beta(I_0)$. These secondary seats will develop new fires in turn. Both aspects refer to some compact dilation δ. We propose to model the seats spread $\beta(I_0)$ by picking out, randomly, a few points in each dilate $\delta(x_i)$, for all points $x_i \in I_0$. The double spread process is then written for the spread of

$$\text{the fire } X_1(I_0) = \delta(I_0) = \bigcup\{\delta(x_i),\ x_i \in I_0\}, \tag{8}$$

$$\text{the seats } I_1(I_0) = \beta(I_0) = \bigcup\{(\delta(x_i) \cap J_i)\,,\ x_i \in I_0,\ J_i \in \mathcal{J}(\theta)\}, \tag{9}$$

where a different realization J_i of the Poisson points process $\mathcal{J}(\theta)$ is associated with each point x_i. Therefore, each point x_i of the set I_0 induces a bunch of seats $\delta(x_i) \cap J_i$ independent of the others. These two equations mean that though the fire from a seat x does burn the zone $\delta(x)$ around x, only a few points of the scar $\delta(x)$ remain active seats for the next step.

Under iteration, Equations 8 and 9 become

$$X_2(I_0) = \delta(I_1) = \bigcup\{\delta(y_k),\ y_k \in I_1\} = \bigcup\{\delta(y_k),\ y_k \in \delta(x_i) \cap J_i\ ;\ x_i \in I_0\},$$

$$I_2(I_0) = \beta(I_1) = \beta[\beta(I_0)] = \bigcup_i\{\bigcup_k[\delta\,(\delta(x_i) \cap J_i)] \cap J_k\},\ x_i \in I_0\}.$$

Figure 2 depicts the first three steps of a random spread, for which:

- the initial seat I_0 is the point x_0, and the first spread, or front, the dark grey disk $X_1(I_0) = \delta(x_0)$;
- then two Poisson points, namely y_1 and y_2, fall in $\delta(x_0)$. They generate

$$X_2(x_0) = \delta(y_1) \cup \delta(y_2) = \delta(I_1), \text{ in medium grey,}$$
$$\text{and } I_1(x_0) = \{y_1\} \cup \{y_2\};$$

- then again a new Poisson realization generates one point, z_1 in $\delta(y_1)$, and another Poisson realization the three points $z_{2,1}, z_{2,2}$, and $z_{2,3}$ in $\delta(y_2)$, hence

$$X_3(x_0) = \delta(z_1) \cup [\delta(z_{2,1}) \cup \delta(z_{2,2}) \cup \delta(z_{2,3})] = \delta(I_2), \text{ in light grey,}$$
$$\text{and } I_2(x_0) = \{z_1\} \cup [\{z_{2,1}\} \cup \{z_{2,2}\} \cup \{z_{2,3}\}].$$

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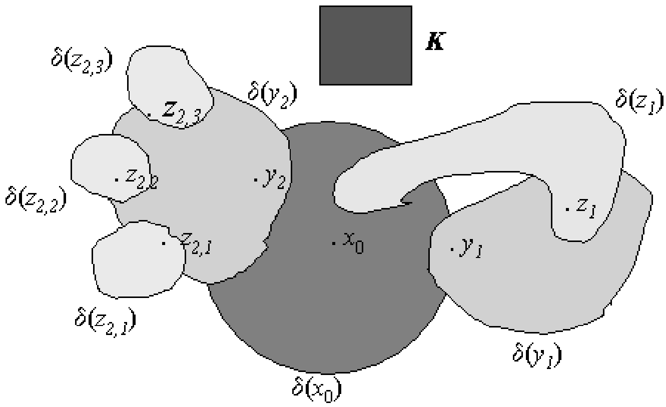

Figure 2. Three generations of fires stemming from point $x_0 = I_0$. Note the generation of new fires in already burnt areas.

The doublet spread (X_n, I_n) of order n, is the n^{th} element of the chain depicted in Figure 3, which is of Markov type: as soon as I_n is known, the previous links do not serve in the creation of link (X_{n+1}, I_{n+1}). This Markov type assumption means that the fire of tomorrow can only be caused by points seats stemming from the zone which burns today. What burnt yesterday, before yesterday, etc., has no longer importance. In this space-time process, the successive sets $X_1, X_2 \ldots X_n$ are at least as descriptive as the seats $I_1, I_2 \ldots I_n$ themselves, because they show the extension of the fire front through the time steps. The $(n+1)$ step is obtained by the two induction relations

$$X_{n+1}(I_0) = \delta[I_n(I_0)] \tag{10}$$

$$\begin{aligned} I_{n+1}(I_0) &= \beta[I_n(I_0)] \\ I_{n+1}(I_0) &= \bigcup\{(\delta(x_{i,n}) \cap J_{i,n}), \quad x_{i,n} \in I_n, \quad J_{i,n} \in \mathcal{J}(\theta)\}. \end{aligned} \tag{11}$$

We observe that, in Equation 9, and the ulterior ones, each small zone dz intervenes as many times as dz belongs to different $\delta(x_i)$. Therefore, the measure

$$\tau_{n+1}(dz) = \theta(dz) \sum \{1_{\delta(x_{i,n})}(z), \quad x_{i,n} \in I_n\} \tag{12}$$

turns out to be a realization of the Cox process of intensity τ_{n+1}, if I_{n+1} is finite. Now, are we sure that all these handlings and derivations really lead to a RACS, in Matheron-Kendall's sense, i.e., to something we can characterize by a functional $Q(K)$? Are we sure, for example, that the intensity τ_{n+1} is always a.s.finite? The answer is positive, as soon as the structuring function δ is *compact*, in the sense of Definition 1. Then one can state the following proposition [13].

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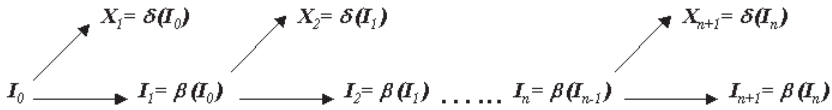

Figure 3. The Markov chain of the spreads.

Proposition 1. *Let I_0 be a a.s. locally finite set of points, let θ be a Poisson intensity, and $\delta : \mathbb{R}^d \to \mathcal{K}(\mathbb{R}^d)$ be a structuring function. If function δ is compact, then both families $\{X_n(I_0), n > 0\}$ and $\{I_n(I_0), n > 0\}$ of fire spreads and seats spreads are RACS.*

Since the two spreads I_n and X_n are RACS, we shall characterize them by their functionals $Q_n(K)$. In the "Boolean-Cox RACS" X_n, the primary grain *only* is independent of step n, since at each time n, the intensity τ_n is a new one. This circumstance simplifies the theoretical study of the time evolution. Also, it suggests to find induction relations between $Q_n(K)$ and $Q_{n+1}(K)$ that reflect the two definitions by induction of Equations 10 and 11.

4.2 Characteristic functional

The additivity property of the random spread allows us to take for I_0 a point initial seat, x_0 say, of dilate $X_1 = \delta(x_0)$, and whose intersection of the dilate with Poisson points J provides the first random set $I_1 = \delta(x_0) \cap J$. The functional $Q_n(K \mid x_0)$ of the random fire spread $X_n(x_0)$, i.e., the probability that set K misses the n^{th} spread $X_n(x_0)$ of initial seat x_0, satisfies an induction relation between steps n and $n+1$. The compact set K lies in the pores of the $(n+1)^{th}$ spread if and only if none of the points $y \in \delta(x_0)$ can develop a n^{th} spread that hits K. For a given $y \in \delta(x_0)$, this elementary probability is

$$dQ_{n+1}(K \mid x_0 \mid y) = 1 - \theta(dy) + \theta(dy)Q_n(K \mid y), \qquad dy \in \delta(x_0). \tag{13}$$

As the events occurring in disjoint dy are independent, we obtain $Q_{n+1}(K \mid x_0)$ by taking the infinite product inside $\delta(x_0)$, i.e.,

$$Q_{n+1}(K \mid x_0) = \exp - \int_{\delta(x_0)} \theta(dy)[1 - Q_n(K \mid y)]. \tag{14}$$

Each step involves an exponentiation more than the previous one. We find for example for the first steps that

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$$Q_2(K) = \exp -\theta[\zeta(K) \cap \delta(x_0)], \tag{15}$$
$$Q_3(K) = \exp -\int_{\delta(x_0)} \theta(dy)[1-e^{-\theta[\zeta(K)\cap\delta(y)]}],$$
$$Q_4(K) = \exp -\int_{\delta(x_0)} \theta(dy)[1-\exp\{-\int_{\delta(y)} \theta(dz)[1-e^{-\int_{\delta(z)}\theta(dw)1_{\zeta(K)}(w)}]\}].$$

where Q_2, but neither Q_3 nor Q_4, is equivalent to the Boolean RACS functional of Equation 7. The seat spread I_{n+1} satisfies the same induction relation 14 as the fire spread X_{n+1}. The only change holds on the first term, for which it suffices to replace $\zeta(K)$ by K in Equation 15. We can summarize the main results on the characteristic functional by stating:

Theorem 2. *Let*
- β be the random spread of parameters (θ, δ),
- $I_1 = \beta(x_0)$ be the random seat spread stemming from point x_0 of dilate $X_1 = \delta(x_0)$,
- $I_2 = \beta(I_1)$ and $X_2 = \delta(I_1)$ be the iterated seat spread and its fire spread,
- $I_{n+1} = \beta(I_n)$ be n^{th} iteration of β, and $X_{n+1} = \delta(I_n)$ the associated fire spread,
then the characteristic functionals of both RACS I_{n+1} and X_{n+1} satisfy the induction relation

$$Q_{n+1}(K \mid x_0) = \exp -\int_{\delta(x_0)} \theta(dy)[1 - Q_n(K \mid y)],$$

with initial terms

$$\begin{aligned} Q_1(K) &= \exp -\theta[K \cap \delta(x_0)] \quad \textit{for the seat spread } I_1 \textit{ and} \\ Q_2(K) &= \exp -\theta[\zeta(K) \cap \delta(x_0)] \quad \textit{for the fire spread } X_2. \end{aligned}$$

4.3 Discussion

A Markov type assumption of order one underlies the random spread model, since it suffices to know the n^{th}seats for constructing the $(n+1)^{th}$ ones. This assumption just allowed us to achieve the formal calculus of the functionals Q_n and R_n. But it does not prevent us from the comeback of new seats on already burnt areas, after a few steps. The successive spreads may turn around $\delta(x_0)$, as depicted in Figure 2.

The trouble can partly be overcome by a Markov assumption of order two, i.e., by making depend the $(n+2)^{th}$ seats of both the $(n+1)^{th}$ and n^{th} steps. We can impose for example to keep the second seats if they fall in the $\delta(x_i), x_i \in I_1$ but *not* in $\delta(x_0)$, and so on. The new version of the

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induction relation 14 must now involve two successive terms. Given the two points y and z the equation 13 of the elementary functional becomes

$$dQ_{n+1}(K \mid x_0 \mid y \mid z) = 1 - \theta(dz) 1_{\delta(y)\setminus\delta(x_0)}(z) + \theta(dz) 1_{\delta(y)\setminus\delta(x_0)}(z) Q_n(K \mid z),$$

which gives, by integration in z

$$Q_{n+1}(K \mid x_0 \mid y) = \exp\left\{ -\int_{\delta(y)\setminus\delta(x_0)} \theta(dz)[1 - Q_n(K \mid z)] \right\}$$

and finally

$$Q_{n+2}(K \mid x_0) = \exp\left\{ -\int_{\delta(x_0)} \theta(dy)[1 - Q_{n+1}(K \mid x_0 \mid y)] \right\}$$

$$= \exp\left\{ -\int_{\delta(x_0)} \theta(dy)[1 - \exp\{-\int_{\delta(y)\setminus\delta(x_0)} \theta(dz)[1 - Q_n(K \mid z)]\}] \right\}.$$

More generally, if the random spread of order n satisfies a Markov assumption of order n, one then obtains a new representation of the hierarchical RACS ([11], p.562, [5], Ch. 6), probably more tractable.

4.4 Spontaneous extinction

The fire which stems from the point seat x_0 may go out, spontaneously, after one, two, or more steps. The description of this phenomenon does not involve any particular compact set K. Denote by $g(n \mid x_0)$ the probability that the fire extinguishes after step n. This event occurs after the first step when no Poisson point falls inside set $\delta(x_0)$, i.e., with the probability

$$g(1 \mid x_0) = 1 - \exp -\theta[\delta(x_0)].$$

The proof by induction that allowed us to link Q_{n+1} with Q_n in Equation 14 applies again, and gives, for a spontaneous extinction after step $n+1$, the probability

$$g(n+1 \mid x_0) = 1 - \exp\{ -\int_{\delta(x_0)} \theta(dy) g(n \mid y)\}. \tag{16}$$

For example, the probability $g(3 \mid x_0)$ of an extinction after the third step is:

$$1 - \exp - \int_{\delta(x_0)} \theta(dy)[1 - \exp\{ -\int_{\delta(y)} \theta(dz)[1 - \exp\{-\int_{\delta(z)} \theta(dw)\}]\}].$$

Suppose for the moment that θ and δ are translation invariant. The extinction probabilities no longer depend on x_0, y, etc., and reduce to $g(n+$

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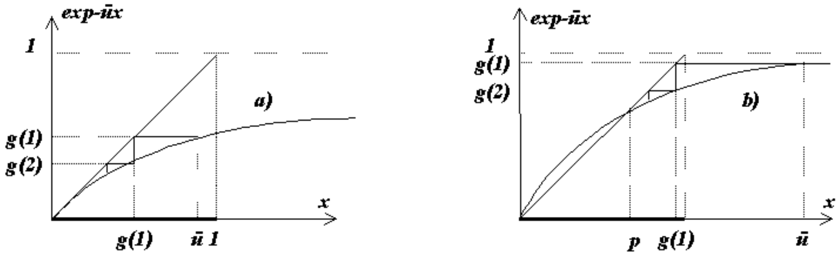

Figure 4. (a) The weight $\bar{u} < 1$, then $g(n) \to 0$, as $n \to \infty$. (b) The weight $\bar{u} \geq 1$, then $g(n) \to p$, as $n \to \infty$.

$1 \mid x_0) = g(n{+}1)$, $g(n \mid y) = g(n)$, etc. Similarly, the integral $\bar{u} = \int_{\delta(z)} \theta(dx)$ is independent of $z \in \mathbb{R}^d$, so that

$$g(n+1) = 1 - \exp -\bar{u}g(n). \tag{17}$$

As $n \to \infty$, the behavior of g depends on $\bar{u}$. If $\bar{u} < 1$, which corresponds to Figure 4(a), then $g(n) \to 0$, i.e., the fire extinguishes spontaneously, almost surely, in a finite time. When $\bar{u} \geq 1$, then the two curves of Figure 4(b) intersect at point (p, p), where $p > 0$ is solution of the equation $p = 1 - \exp \bar{u}p$. There is a non zero probability, namely p, of an infinite spread.

Suppose now that both functions θ and δ vary over the space, and let Z be the set of all points where $u(z) \geq 1$. If $x_0 \in Z$, then there is every chance that the fire invades the connected component of $\delta(Z)$ that contains point x_0. Such typical behaviour will be used now for predicting the scars of forest fires in the State of Selangor.

5. Scar prediction

The forestry services call *scar* of the fire the cumulative process

$$Y(x_0) = \bigcup \{X_p(x_0), 1 < p < \infty\}. \tag{18}$$

We will now match the data of the actual scars with the model. Section 4.4 has shown the crucial role played by the weight $u\,(x)$. In each region Z where all $u\,(x)$, $x \in Z$, are noticeably ≥ 1, any initial seat invades progressively the whole region, whereas in the regions with $u\,(x) < 1$, the spread stops by itself, all the sooner since $u\,(x)$ is small. In Selangor's case, the expression of u from the two maps of Figure 1 is as follows

$$u\,(x) = \int \theta\,(dz)\, 1_{\zeta(x)}\,(z) = k \int f_w\,(z)\, 1_{\zeta(x)}\,(z)\, dz \simeq \pi k f_w\,(x)\, r^2\,(x)\,.$$

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This expression suggests to introduce the *scar function*

$$s(x) = f_w(x)\, r^2(x)\,,$$

as represented in Figure 5. Scar function s is accessible from the experimental data, since functions f_w and r are given (see Figures 1(c) and 1(a)). Note that map s is not obtained by simulations, but comes from a combination of the input parameters of the random spread model. By putting a threshold on image s at level $1/\ \pi k$, one splits the plane into the two regions where, either fires spontaneously extinguish (when $s(x) < 1/\ \pi k$), or invade the connected components that contain their initial seats (when $s(x) \geq 1/\ \pi k$).

If we take for k the value 1.61×10^{-3}, which derives from the hot spots measurements [16], we get $1/\ \pi k = 193$. The two sets above thresholds 190 and 200 are reported in Figure 5(b), side by side with the burnt areas (Figure 5(c)). In Figure 5(b), the fire locations **A** to **E** predicted by the model point out regions of actual burnt scares. Such a remarkable result could not be obtained from the maps f_w and r taken separately: the scare function $s = f_w r^2$ means something more, which corroborates the random spread assumption. Region **F** is the only one which seems to invalidate the model. As a matter of fact this zone is occupied by peat swamp forest, or rather, *was occupied.* It is today the subject of a fast urbanization, linking the international airport of Kuala Lumpur to the administrative city of Putra Jaya. Finally, on the whole, the random spread model turns out to be realistic.

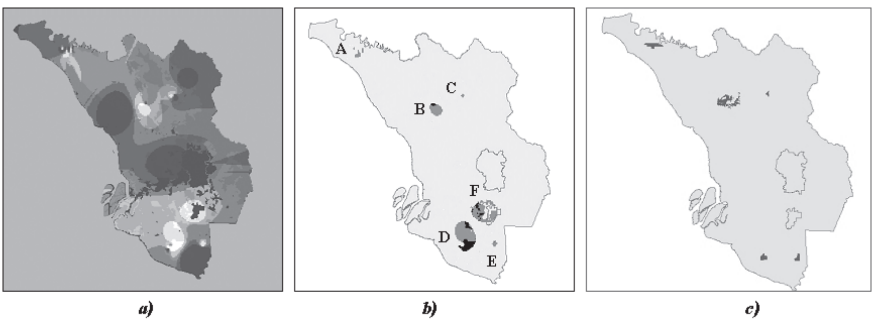

Figure 5. (a) Scar function $s = f_w \times r^2 = u/\pi k$ whose thresholds estimate the burnt scar zones. (b) Two thresholds of function s for $1/\pi k = 190$, in dark grey, and $1/\pi k = 200$, in black (the simulations suggest value 193). (c) Map of the actual burnt areas. Note the similarity of the sets, and of their locations.

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6. Conclusion

This paper proposes a new RACS model, the *random spread*, which combines the three theoretical lines of Boolean random sets, Markov chains, and birth-and-death processes. Its characteristic functional was established. More than classical spatial birth-and-death processes, spread RACS depends strongly on the heterogeneity of the space, which appears via two functions of intensity (θ) and extension (δ). As a result, the process no longer describes a global birth-and-death, but regional expansions and shrinkages of the sets under study, namely the front, the seats and the scar of the spread RACS.

The time evolution was introduced in a discrete manner, by the Markov assumption that the fire front of tomorrow can only be caused by points seats stemming from the zone which burns today.

We draw from the model and a precise predictor of scars that actually occurred in the State of Selangor during the period 2000–2004.

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